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ON UNBIASED LEHMANN - ESTIMATORS OF A VARIANCE OF AN EXPONENTIAL DISTRIBUTION WITH QUADRATIC LOSS FUNCTION

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SUMMARY

Lehmann in [4] has generalised the notion of the unbiased estimator with respect to the assumed loss function. In [5] Singh considered admissible estimators of function λ^{-r} of unknown parameter λ of gamma distribution with density $f(x \mid \lambda, b) = \lambda^{b-1} e^{-\lambda x} x^{b-1}/\Gamma(b)$, $x > 0$, where b - known parameter, for loss function $L(\hat{\lambda}^{-r}, \lambda^{-r}) = (\hat{\lambda}^{-r} - \lambda^{-r})^2/\lambda^{-2r}$.

Goodmann in [1] choosing three loss functions of different shape found unbiased Lehmann-estimators, of the variance σ^2 of the normal distribution. In particular for quadratic loss function he took weight of the form $K(\sigma^2) = C$ and $K(\sigma^2) = (\sigma^2)^{-2}$ only.

In this work we obtained the class of all unbiased Lehmanns-estimators of the variance λ^2 of the exponential distribution, among estimators of the form $\alpha(n) (\sum_1^n X_i)^2$ – i e functions of the sufficient statistics - with quadratic loss function with weight of the form $K(\lambda^2) = C(\lambda^2)^{C_1}$, $C > 0$.

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RESUMEN

Lehmann en su trabajo [4] generalizó la idea del estimador sin vías en relación a la aceptación de la función de pérdidas. En el trabajo [5] Singh considera de estimadores admisibles para la función λ^{-r} parámetro desconocido λ de la distribución de gamma, de densidad $f(x \mid \lambda, b) = \lambda^{b-1} e^{-\lambda x} x^{b-1} / \Gamma(b)$, $x > 0$, $b > 0$ parámetro conocido, de la función de pérdidas resulta $L(\hat{\lambda}^{-r}, \lambda^{-r}) = (\hat{\lambda}^{-r} - \lambda^{-r})^2 / \lambda^{-2r}$.

Goodman en su trabajo [1] acumulando 3 formas diferentes de funciones de pérdidas encontró estimadores sin biases en el sentido de Lehmann de la variancia de σ^2 de una distribución normal, en particular para la función de pérdidas $L(\hat{\sigma}^2, \sigma^2) = K(\sigma^2)\ (\hat{\sigma}^2 - \sigma^{-2})^2$ con los pesos, solo de la forma $K(\sigma^2) = C$, $K(\sigma^2) = (\sigma^2)^{-2}$.

En su trabajo distinguida la clase de todos los estimadores sin biases obtenidos en el sentido de Lehmann de la variancia λ^2 en la distribución exponencial, entre los estimadores de forma $\alpha(n)\ (\sum_1^n X_i)^2$ –así pues de la función estadística suficiente– por una función de pérdidas al cuadrado con los pesos de la forma $K(\lambda^2) = C(\lambda^2)^{C_1}$, $C > 0$.

Palabras y frases. Estimador sin vías en el sentido de Lehmann, función de pérdidas, riesgo mínimo, suficiente estadística.

1. INTRODUCTION

In some problems concerned to an estimation of the parameter on which depends probability distribution of searched random variable there is essential a question about an error arisen from replacement of true, but unknown, value of the parameter θ by its estimator $\hat{\theta}$. There is assumed that, so called, loss function $L(\hat{\theta}, \theta)$ is an estimate of that error. The loss function can be of different form dependently on the purpose.

Assuming, that θ is a real value of a parameter, the conditional expected value of a loss function defines the risk $R(\theta) = E\ [L(\hat{\theta}, \theta) \mid \theta]$ of the estimator $\hat{\theta}$.

In the article there are given estimators (defined univalently in the considered class of estimators) of a parameter λ^2, i.e., variance of the distribution

$$f_X(x \mid \lambda) = \begin{cases} 0 & x \leqslant 0 \\ \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right) & x > 0 \end{cases} \qquad (1)$$

minimizing the risk by defined loss function and estimators unbiased with respect to the assumed form of a loss function.

The definition of an estimator unbiased with respect to a given form of a loss function was introduced by Lehmann in the article [4]:

The estimator $\hat{\theta}$ of a parameter θ is called unbiased with respect to the loss function $L(\hat{\theta}, \theta)$ if for every value of a parameter $\theta \in \Theta$ the risk $R(\theta_1) = E[L(\hat{\theta}, \theta_1) \mid \theta]$ attains a minimum for $\theta_1 = \theta$.

Using the Lehmann's definition in the elementary way one can prove the equivalence of the unbiasedness in the usual sense of the estimator $\hat{\theta}$ and unbiasedness with respect to the quadratic loss function

$$L(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2, \quad \text{where} \quad c \in R_+$$

The density function (1) belongs to one-parameter class of distributions of exponential type. It follows' from [6, §2, 4] that minimal sufficient statistics for this class of distributions is the statistics.

$$T = \sum_{1}^{n} X_i \qquad (2)$$

where X_i $(i = 1, 2, ..., n)$ are independent random variables, each of which has the distribution (1).

The notion of minimal sufficient statistic is not uniquely determined, what is well known, for each invertible function of minimal sufficient statistics is also minimal sufficient statistics.

We have $E(X) = \lambda$ for random variable with the distribution (1). Because of it, the statistics.

$$Z = \frac{1}{n} \sum_{1}^{n} X_i \qquad (3)$$

is an unbiased estimator of the parameter λ of the distribution (1). Since for considered random variable $\text{Var}\, X = \lambda^2 = (EX)^2$, then as a minimal sufficient statistics, from which we make dependent seeking estimators $\hat{\lambda}^2$ of the variance λ^2, we will assume

$$Y = (\sum_{1}^{n} X_i)^2 \qquad (4)$$

which is a function of a minimal statistics (2) sufficient for oneparameter class of distributions (1); the invertibility of the function (4) is assured by the condition (2) with $X_i \geqslant 0$.

Finally, the class of estimators of the variance λ^2 of the distribution (1) we define as follows

$$\hat{\lambda}^2 = \alpha Y \qquad (5)$$

where α is a positive function defined on the set of positive integers.

In the following we will use the lemma, which is easy to prove.

Lemma. If each of independent random variables X_i $(i = 1, ..., n)$ has the distribution with the density function $h(x_i \mid \lambda) = \frac{1}{\lambda} f \frac{x_i}{\lambda}$ what means that λ is the parameter of the scale, then

$$\beta = \frac{\lambda^2 E(Y \mid \lambda)}{E(Y^2 \mid \lambda)}$$

where Y defined by (4) is independent from λ for a fixed n.

2. Unbiased estimator of the variance λ^2 of the exponential distribution (1).

Because of the parameter λ of the exponential distribution (1) is a scale parameter and the statistics (4) is homogenens of the order 2 with respect to variables X_i ($i = 1, ..., n$), then we use the theorem [6: §3.6], which establishes the form of an unbiased estimator for an arbitrary power of a scale parameter in the class of homogenens statistics of an arbitrary order $r \neq 0$. Since we estimate λ^2, then

$$\hat{\lambda}^2 = \frac{1}{n\,(n+1)} \left(\sum_1^n X_i\right)^2 \tag{6}$$

is an unbiased estimator in the class (5).

It is easily to see, that this estimator is consistent. Since the statistics (3) is an effective estimator of the parameter λ of the distribution (1), then, what is well known [3: § 17], there exists no effective estimator for any other function of the parameter λ, thus for λ^2 as well.

Now it is worth to remark that the estimator (6) has another important property. We know, that the statistics (3) is sufficient for the class of distributions (1). From the form of a density function of this statistics we infer, on a base of Lehmann's theorem [2: §7.1], that one-parameter class of distributions of the exponential type of the form (1) is a complete class with respect to the sufficient statistic (3), and for complete class every function $t(\theta)$ of an unknown parameter θ permits no more than one unbiased estimator, which is a function of the sufficient statistics for this class.

According to this considerations and the theorem [2, th, 7.1.2] we infer that the statistics (6) is an optimal estimator of the function $t(\lambda) = \lambda^2$ in the class of all unbiased estimators of the variance λ^2 by an arbitrary loss function which is convex with respect to an estimator.

Remark 1. If we assume a quadratic loss function of the form $k(\lambda^2)\,(\hat{\lambda}^2 - \lambda^2)^2$ with the weight $k(\lambda^2) = 1$ then for a given unbiased estimator the risk will be its variance. Since the estimator (6) is an optimal one in the class of estimators unbiased with respect to an arbitrary loss function which is convex with respect to the estimator, then it is an estimator with a minimal variance in the class of unbiased estimators.

Of course, there exist the other estimators of the variance λ^2, unbiased in the usual sense, which are not still the functions of sufficient statistics,for example $\frac{1}{2n}\sum_1^n X_i^2$.

Remark 2. It follows from an introduced in [2] optimality and admissibility of estimator in a given class of estimators with a fixed loss function, that every optimal estimator is an admissible estimator in the class of estimators.

In this way the estimator (6) is an admissible estimator of the variance of the distribution (1) in the class of estimators of the form (5), with an arbitrary loss function which is convex with respect to the estimator.

R. Singh [5] studying estimators of a function of an unknown parameter λ in the gamma distribution of the form.

$$f_1(x \mid \lambda) = \begin{cases} 0 & x \leqslant 0 \\ \dfrac{\lambda^b}{\Gamma(b)} e^{-\lambda x} x^{b-1} & x > 0 \end{cases}$$

with the known positive parameter b, found the form of admissible estimator for the function $q(\lambda) = \lambda^{-r}$($r$ *is* an integer) with the loss function of the form:

$$L(\hat{\lambda}^{-r}, \lambda^{-r}) = \left[\frac{\hat{\lambda}^{-r} - \lambda^{-r}}{\lambda^{-r}}\right]^2$$

3. Estimators of a variance λ^2 with a quadratic loss functions.

Let the loss function be of the form

$$L(\hat{\lambda}^2, \lambda^2) = K(\lambda^2)\,(\hat{\lambda}^2 - \lambda^2)^2 \tag{7}$$

We assume that the function $K(\lambda^2)$, called a weight, is an arbitrary positive differentiable function.

The following theorem will be very useful.

Theorem 1. If the class of estimators of the variance of the distribution (1) is of the form (5) and the loss function is of the form (7), then the only estimator $\alpha_0 Y$ where

$$\alpha_0 = \frac{1}{(n+2)(n+3)} \tag{8}$$

minimizes the risk $R(\alpha | \lambda^2) = E\,[L(\hat{\lambda}^2, \lambda^2) | \lambda]$.

The easy proof of this theorem is omited. From the theorem 1 it follows that the estimator

$$\tilde{\lambda}^2 = \frac{1}{(n+2)(n+3)} (\sum_1^n X_i)^2 \tag{9}$$

is an estimator of the variance of the distribution (1) minimizing, for every value λ^2, the risk by the loss function (7) with an arbitrary but fixed positive weight $K(\lambda^2)$.

Corollary 1.1. The statistics (9) is an optimal estimator of the variance of the distribution (1) in the class of estimators (5) with the loss function (7).

In spite of estimators (6) and (9) belong to the class of estimators (5), they are optimal in different classes. So, for an arbitrary unbiased estimator $\hat{\hat{\lambda}}^2 \neq \hat{\lambda}^2$ (where $\hat{\lambda}^2$ is an estimator of the form (6)) with an arbitrary loss function, which is convex with respect to the estimator,

$$R(\hat{\lambda}^2 | \lambda^2) \leqslant R(\hat{\hat{\lambda}}^2 | \lambda^2)$$

holds for every value $\lambda > 0$.

Next, for an arbitrary estimator $\tilde{\tilde{\lambda}}^2 \neq \tilde{\lambda}^2$ from the class (5) the inequality

$$R(\tilde{\lambda}^2 | \lambda^2) \leqslant R(\tilde{\tilde{\lambda}}^2 | \lambda^2)$$

is true for every value $\lambda > 0$, thus the estimator (9) minimizes the risk in the class of estimators of the form (5) with the loss function of the form (7).

Since both of considered estimators belong to the class (5), then with the loss function (7) the relation

$$R(\tilde{\lambda}^2 \mid \lambda^2) < R(\hat{\lambda}^2 \mid \lambda^2)$$

holds for every value $\lambda > 0$.

Corollary 1.2. The estimator (9) is a consistent estimator of the variance λ^2 of the distribution (1).

Since, according to the considerations of § 2, the unbiased estimator of an unknown parameter, which is the function of a sufficient statistics, is uniquely determined in the complete class, then the estimator (9) is not unbiased one in the usual sense.

Then there arises a natural question: Is it possible for this estimator to be unbiased with respect to the loss function of the form (7) by the adequately chosen weight $K(\lambda^2)$ of the squared loss function?

Theorems 2 and 3 give the answer for this question.

Theorem 2. If there exists in the class (5) an estimator of the variance λ^2 of the distribution (1) which is unbiased with respect to the loss function of the form (7) and $\alpha = \alpha_0$ is given by the formula (8), then the only function:

$$K(\lambda^2) = \frac{C}{(\lambda^2)^2} \quad C \in R_+ \tag{10}$$

can be the weight.

Proof. After the differentiation the risk $R(\lambda_1^2 \mid \lambda^2, \alpha_0)$ with respect to λ_1^2 we infer

$$\frac{\partial R(\lambda_1^2 | \lambda^2, \alpha_0)}{\partial \lambda_1^2} = \frac{dK(\lambda_1^2)}{d\lambda_1^2} \lambda^4 \left[\frac{n(n+1)}{(n+2)(n+3)} - \right.$$

$$\left. - 2 \frac{n(n+1)}{(n+2)(n+3)} \cdot \frac{\lambda_1^2}{\lambda^2} + \left(\frac{\lambda_1^2}{\lambda^2} \right)^2 \right] +$$

$$+ 2 \lambda^2 K(\lambda_1^2) \left[\frac{\lambda_1^2}{\lambda^2} - \frac{n(n+1)}{(n+2)(n+3)} \right]$$

Since the estimator $\alpha_0 Y$ is unbiased with respect to the loss function of the form (7), then for an arbitrary value of the parameter λ^2 the risk attains minimum for $\lambda_1^2 = \lambda^2$, then the calculated derivative equals to zero if $\lambda_1^2 = \lambda^2$. Then we obtain the differential equation

$$\frac{dK(\lambda^2)}{d\lambda^2} = - \frac{2 K(\lambda^2)}{\lambda^2}$$

the general solution of which is the function (10).

Theorem 3. In the class of estimators of the form (5) of the variance of the distribution (1), the only unbiased estimator with respect to the loss function (7) with the weight (10) is the estimator $\alpha_0 Y$ where α_0 is defined by (8).

Proof. The derivative of the risk $R(\lambda_1^2, \alpha | \lambda^2)$ with the loss function (7) and weight (10) is equal to

$$\frac{\partial R(\lambda_1^2, \alpha | \lambda^2)}{\partial \lambda_1^2} = \frac{2 C(\lambda^2)^2}{(\lambda_1^2)^3} \alpha \left[\frac{\lambda_1^2}{\lambda^2} - \alpha (n+2)(n+3) \right] n(n+1)$$

From its form we infer that this derivative equals to zero, and the risk attains the minimum, for $\lambda_1^2 = \lambda^2$ if and only if

$$\alpha = \frac{1}{(n+2)(n+3)} = \alpha_0$$

This ends the proof.

Corollary 2.1. The risk responsive to the estimator (9) with the loss function (7) does not depend on true value of the parameter λ if and only if $K(\lambda^2) = \frac{C}{(\lambda^2)^2}$.

Theorem 1 decides that the estimator (9), and only this one, minimizes the risk with quadratic loss function (7) by an arbitrary but fixed weight $K(\lambda^2)$.

Theorems 2 and 3 imply that this estimator is unbiased with respect to the loss function of the form (7) with the weight (10). Thus another important problem arises: Are there, for some other weights $K(\lambda^2)$, any estimators unbiased with respect to the new loss functions? It is worth to remind that Goodman found in the work [1] the unbiased estimator of variance σ^2 in the normal distribution $N(\mu, \sigma^2)$ (μ-unknown) with respect to the quadratic loss function, but with the weight $(\sigma^2)^{-2}$ (analogously to $(\lambda^2)^{-2}$).

Theorem 4. If there exists in the class (5) an estimator of the variance of the exponential distribution (1) which is unbiased with respect to the quadratic loss function of the form (7) and C_1 is such a constant value that the equation of α

$$n(n+1)(n+2)(n+3)\,C_1\,\alpha^2 - 2n(n+1)(C_1+1)\alpha + C_1 + 2 = 0 \quad (11)$$

has a positive solution, then the function

$$K(\lambda^2) = C(\lambda^2)^{C_1}, \quad C \in R_+ \quad (12)$$

can only be the weight of the loss function (7).

Proof. The risk $R(\lambda_1^2 \mid \lambda^2, \alpha)$ with the loss function (7) is equal to

$$R(\lambda_1^2 \mid \lambda^2, \alpha) = K(\lambda_1^2)\,[n(n+1)(n+2)(n+3)\lambda^4\alpha^2 - \\ - 2n(n+1)\lambda^2\lambda_1^2\alpha + (\lambda_1^2)^2]$$

Differentiating it with respect to λ_1^2 we infer

$$\frac{\partial R(\lambda_1^2|\lambda^2,\alpha)}{\partial\lambda_1^2}=\frac{dK(\lambda_1^2)}{d\lambda_1^2}[n\,(n+1)\,(n+2)\,(n+3)\,\lambda^4\,\alpha^2-$$

$$-2\,n\,(n+1)\,\lambda^2\,\lambda_1^2\,\alpha+(\lambda_1^2)^2]+K(\lambda_1^2)\,[2\,\lambda_1^2-2\,n\,(n+1)\,\alpha\,\lambda^2]$$

From unbiasedness with respect to the loss function of the form (7) it follows that for every value λ^2 this risk attains a minimum for $\lambda_1^2=\lambda^2$ and simultaneosly for $\lambda_1^2=\lambda^2$ the considered derivative equals to zero, what implies the differential equation

$$\frac{1}{K(\lambda^2)}\,\frac{dK(\lambda^2)}{d\lambda^2}=\frac{2\,n\,(n+1)\,\alpha-2}{n\,(n+1)\,(n+2)\,(n+3)\,\alpha^2-2\,n\,(n+1)\,\alpha+1}\cdot\frac{1}{\lambda^2}$$

where the denominator of the second fraction on the right side in the last equality is positive for all α. Since from assumption, α is a positive solution of the equation (11), then the first fraction on the right side in that equality is equal to C_1. Next, it is easily to prove, that the general solution of the obtained differential equation is of the form (12).

We have yet to prove that there exists a constant C_1 such that the equation (11) has a proper (it means - positive) solution.

Let us consider some cases.

a) $C_1=0$. The equation (11) is in this case of the first order, and the solution of it is of the form

$$\alpha'=\frac{1}{n\,(n+1)}$$

The estimator $\alpha'\,Y$ from the class (5), discussed in §2, is an unbiased (in the usual sense) estimator of the variance λ^2 of the exponential distribution (1), hence it is unbiased with respect to the quadratic loss function with a constant weight.

b) $C_1=-2$. Then $\alpha=0$ (we reject it) and $\alpha=\alpha_0$ defined by (8) are the only solutions of the equation (11). Hence according to the theorem 3 the estimator $\alpha_0 Y$ is unbiased with respect to the loss function of the form (7) with the weight (10).

c) $C_1 \neq 0$, $C_1 \neq -2$. Then the discriminant of the equation (11) equals to

$$\Delta = -4n(n+1)[2(2n+3)C_1^2 + 4(2n+3)C_1 - n^2 - n] \quad (13)$$

It is no negative if and only if

$$|C_1 + 1| \leqslant \sqrt{\frac{n^2 + 5n + 6}{.4n + 6}}$$

Then $\sqrt{1.2}$ is an infimum of the set of numbers of the form $\sqrt{\frac{n^2 + 5n + 6}{4n + 6}}$, where n is an arbitrary positive integer, and this infimum is attained if $n = 1$. Thus the equation (11) has two real solutions for every $n \in N$ if C_1 belongs to the union of intervals

$$\langle -1 - \sqrt{1.2}, -2), \quad (-2, 0), \quad (0, -1 + \sqrt{1.2}\rangle \quad (14)$$

It is easy to be proved, that with such a constant C_1 the equation (11) has got either positive solutions (the first and third interval) or the solutions are of distinct signs (the secoud interval).

In this way the theorem is proved.

Theorem 5. If a loss function is a quadratic function with the weight $K(\lambda^2) = C(\lambda^2)^{C_1}$, where $C \in R_+$ and C_1 is a constant value from one of the intervals (14) then only the estimator $\alpha_1 Y$, where

$$\alpha_1 = \frac{C_1 + 2}{n(n+1)(C_1+1) + \sqrt{n(n+1)[n^2+n-4(2n+3)C_1-2(2n+3)C_1^2]}} \quad (15)$$

is an unbiased one in the class (5) with respect to the assumed loss function.

Proof. After easy calculations we can write that

$$\frac{\partial R(\lambda_1^2, \alpha | \lambda^2)}{\partial \lambda_1^2} = C(\lambda^2)^2 (\lambda_1^2)^{C_1 - 1} \quad (C_1 + 2) \left[\frac{\lambda_1^2}{\lambda^2}\right]^2 -$$

$$-2n(n+1)(C_1+1)\frac{\lambda_1^2}{\lambda^2} + n(n+1)(n+2)(n+3)\alpha^2 C_1]$$

We seek such an α that the risk attains a minimum for $\lambda_1^2 = \lambda^2$. The discriminant of the quadratic polynomial with respect to λ_1^2/λ^2 in the last squared bracket is

$$\Delta_1 = \alpha^2 [4n^2(n+1)^2(C_1+1)^2 - 4n(n+1)(n+2)(n+3)(C_1+2)C_1]$$

and using (13)

$$\Delta_1 = \alpha^2 \Delta$$

From the form of the derivative we infer, in an easy way, taking into consideration the value C_1 from suitable intervals, that the risk $R(\lambda_1^2, \alpha | \lambda^2)$ attains a minimum for

$$\frac{\lambda_1^2}{\lambda^2} = \frac{\alpha [2n(n+1)(C_1+1) + \sqrt{\Delta}]}{2(C_1+2)}$$

for an arbitrary but fixed $\alpha > 0$.

This minimum is attained for $\lambda_1^2 = \lambda^2$ if and only if the right side of the last equality is equal to 1, and next if and only if $\alpha = \alpha_1$, where α_1 is given by (15).

Remark 1. It is easily to verify that the coefficient (15) is a solution of the considered equation (11).

Remark 2. For each C_1 belonging to one of the intervals (14) the right side of (15) is positive.

Remark 3. For an arbitrary C_1 from one of the intervals (14) the estimator of the form

$$\hat{\lambda}^2 = \alpha_1 \left(\sum_1^n X_i\right)^2 \tag{16}$$

where α_1, given in the formula (15), is unbiased with respect to a quadratic loss function with the weight $K(\lambda^2) = C(\lambda^2)^{C_1}$, $C \in R_+$.

Corollary 5.1. The statistics (16) is a consistent and assymptotically unbiased estimator of the variance of the distribution (1).

The results of §2 and theorem 3 and 5 imply that for every $C_1 \in \langle -1 - \sqrt{1.2}\,; -1 + \sqrt{1.2} \rangle$ we can determine an estimator of the variance of the exponential distribution (1) in the class (5) which is unbiased with respect to the quadratic loss function with the weight $K(\lambda^2) = C(\lambda^2)^{C_1}$, $C \in R_+$. For each of those estimators there is adjusted a risk; it is worth to convince oneself, whether such a weight exists, it means whether there exist such a constant C_1 and adequate to it α_1 such that for an arbitrary but fixed value λ^2 the risk corresponding to this estimator $\alpha_1 Y$, which is unbiased with respect to the quadratic loss function with this weight, is minimal.

The interesting of us risk is of the form

$$R(C_1 \mid \lambda^2) = \begin{cases} \dfrac{C(4n+6)}{(n+2)(n+3)} & C_1 = -2 \\ (\lambda_1^2)^{C_1+2}\,[\alpha^2(C_1)\,n(n+1)(n+2)(n+3) - \\ - 2\,\alpha(C_1)\,n\,(n+1) + 1]\,C & C_1 \text{ – defined by (14)} \\ & \alpha(C_1) \text{ – defined by (15)} \\ \dfrac{C(4n+6)}{n(n+1)}\lambda^4 & C_1 = 0 \end{cases}$$

It is not difficult to be proved, that the function $R(C_1 | \lambda^2)$ is continuous at points $C_1 = 0$ and $C_1 = -2$.

Assume now that $C_1 \in \langle -1 - \sqrt{1.2}; -1 + \sqrt{1.2} \rangle$.

The derivative of the function $R(C_1 | \lambda^2)$ is

$$\frac{\partial R(C_1 | \lambda^2)}{\partial C_1} = C(\lambda^2)^{C_1+2} \Big\{ \ln \lambda^2 \, [\alpha^2(C_1)\, n\,(n+1)(n+2)(n+3) -$$

$$- 2\,\alpha(C_1)\, n\,(n+1) + 1] + 2\,\frac{d\alpha(C_1)}{dC_1}\, n\,(n+1)(n+2)(n+3) \cdot$$

$$\cdot \left[\alpha - \frac{1}{(n+2)(n+3)} \right] \Big\} \qquad (17)$$

Now we verify the necessary condition for an existence of an extremum: If C_1 is such a constant value, for which the risk $R(C_1 | \lambda^2)$ attains a minimum for an arbitrary value λ^2, then C_1 cannot depend on λ^2, thus it has to be

$$\begin{cases} \alpha^2(C_1)\, n\,(n+1)(n+2)(n+3) - 2\,\alpha(C_1)\, n\,(n+1) + 1 = 0 \\ \dfrac{d\alpha(C_1)}{dC_1} \left[\alpha(C_1) - \dfrac{1}{(n+2)(n+3)} \right] = 0 \end{cases}$$

Since the discriminant in the first equation $\Delta = -4\,n\,(n+1)(4\,n+6) < 0$ for every C_1, then there exists no C_1 such that the first equality is fulfilled.

In this way there exists no weight of the form $K(\lambda^2) = C(\lambda^2)^{C_1}$ of the quadratic loss function such that the risk - for every but fixed value λ^2 – corresponding to the estimator, which is unbiased with respect to this form of the loss function, is minimal in the class of all risks corresponding to all other unbiased estimators with respect to the quadratic loss functions with the other weights of considered form. Considering the first component and both factors of the second component in the sum in the main brackets of the equality (17) we infer.

I $\quad \alpha^2(C_1)n(n+1)(n+2)(n+3)-2\alpha(C_1)n(n+1)+1>0$

for every C_1 from $<-1-\sqrt{1.2}, -1+\sqrt{1.2}>$,

$$\text{II}\quad \alpha(C_1)-\frac{1}{(n+2)(n+3)}\begin{cases}<0 \text{ for } C_1\in<-1-\sqrt{1.2};-2)\\ =0 \text{ for } C_1=-2\\ >0 \text{ for } C_1\in(-2,-1+\sqrt{1.2}>;\end{cases}$$

$$\text{III}\quad \frac{d\alpha(C_1)}{dC_1}\begin{cases}>0 \text{ for } C_1\in<-1-\sqrt{1.2};-2)\cup(-2;-1+\sqrt{1.2}>\\ =0 \text{ for } C_1=-2\end{cases}$$

Taking into consideration the relations I, II, III we can make sure of the sign of the derivative of the risk function $R(C_1 | \lambda^2)$, and simultaneously of the monotonicity of this function in following cases:

a) $\lambda\in(0,1)$ and $C_1\in<-1-\sqrt{1.2},-2>$. Then $\frac{\partial R(C_1|\lambda^2)}{\partial C_1}<0$, what implies that the risk $R(C_1|\lambda^2)$ is a decreasing function.

b) $\lambda\in(1,\infty)$ and $C_1\in<-2;-1+\sqrt{1.2}>$. In this case $\frac{\partial R(C_1|\lambda^2)}{\partial C_1}>0$ and the risk $R(C_1|\lambda^2)$ is an increasing function.

From above relations if follows that in the case a) the risk attains a minimum at the right end-point of the interval $<-1-\sqrt{1.2}, -2>$, but in the case b) at the left and-point of the interval $<-2, -1+$ $+\sqrt{1.2}>$.

It means that in both cases the required minimum (being the global minimum) is attained at $C_1=-2$.

Because of the relation between C_1 and $\alpha(C_1)$ is one-to-one we infer that for the estimator (9) corresponds a minimal risk in the class of unbiased estimators with respect to the quadratic loss function with weights of the form $K(\lambda^2)=C(\lambda^2)^{C_1}$, where $C\in R_+$ with values C_1 and λ mentioned in cases a) or b).

4. Conclusion

We shall supplement the considerations received up to now with the geometric interpretation of two notions used in the article: i.e. the estimator which minimizes the risk by a given loss function and unbiased estimator with respect to an assumed form of the loss function.

In the first case we can consider the risk $R(\alpha, \lambda^2)$ as a function of two variables α, λ^2, hence the geometric image of the risk by definite sample size is like some surface cantained in the first eighth part of the coordinate system α, λ^2, $R(\alpha, \lambda^2)$, which has the following property: every cutting of that surface by the plane $\lambda^2 = K$, $k > 0$ is a curve attaining always a minimum for the value $\alpha = \alpha_0$ defined by the formula (8), which is connected with the estimator minimizing the risk with respect to the quadratic loss function with any weight in the class of estimators of the form (5).

The geometric interpretation of the unbiasedness in the Lehmann's sense is as follows. If the estimator αY is unbiased with respect to a given loss function, then we can consider the risk $R(\lambda_1^2, \lambda^2) = E[L(\alpha Y, \lambda_1^2 | \lambda]$, which is a function of two variables λ_1^2, λ^2 with a fixed size of n and results of sample. We consider now the surface in the first eight part of the coordinate system $\lambda_1^2, \lambda^2, R(\lambda_1^2, \lambda^2)$, characterizing itself in the way such that every cutting by the plane $\lambda^2 = K$, $K > 0$ is a curve which attains a minimum (perhaps different for different k) lying always an the bisecting plane of the first eight part of the coordinate system $\lambda_1^2, \lambda^2, R(\lambda_1^2, \lambda^2)$; it means on the plane $\lambda_1^2 = \lambda^2$.

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