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RANK ORDER STATISTICS FOR UNEQUAL SAMPLES

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1. Introduction and Relevant Statistics

Suppose $F_n(x)$ and $G_n(x)$ are the empirical distribution functions of two independent samples $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ drawn respectively from populations with continuous distribution functions $F(x)$ and $G(x)$. Employing complicated methods Smirnov derived in 1939 the probability distribution of the statistic

$$D^+_{n,n} = \operatorname*{Sup}_x \, [F_n(x) - G_n(x)]$$

under the null hypothesis $F(x) \equiv G(x)$. Gnedenko and Korolyuk found the probability distribution of $D^+_{n,n}$ by using the random walk model or the geometric theory of paths. In 1960 Reimann and Vincze [2] considered the case of two unequal samples and derived the distribution of the statistic

$$D^+_{m,n} = \operatorname*{Sup}_x \, [m\, F_m(x) - n\, G_n(x)]$$

under the same null hypothesis. For equal sample sizes, Sujan [3] obtained in 1971 the joint distribution of $D^+_{n,n}$ and $R_{n,n}$, the total number of runs of x's and y's. Here we obtain in generalization of her results the joint probability distribution of the rank order statistics $D^+_{m,n}$ and $R_{m,n}$ by using combinatorial methods.

Given two samples $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_n)$ of sizes m and n $(\leqslant m)$ with observations in ascending order of magnitude we pool them and associate a random variable θ_i with the ith observation such that

$$\theta_i = \begin{cases} +1, & \text{if the ith observation is an } x \\ -1, & \text{if the ith observation is a } y \end{cases}$$

Define $S_0 \equiv 0$ and $S_i \equiv \theta_1 + \theta_2 + \cdots + \theta_i$.

If the points (i, S_i) are joined one with the next by straight segments, we obtain a graphical representation of the sequence $\{S_i\}$. Denoting by $R_{m,n}$ the total number of runs of (+ 1)'s and (− 1)'s, our object is to determine

$$P\{D^+_{m,n} \leqslant t, R_{m,n} = \rho\}; \quad t \geqslant m-n, \quad 2 \leqslant \rho \leqslant 2n+1 \tag{1}$$

Now $D^+_{m,n} \leqslant t$, if $\max\limits_{1 \leqslant i \leqslant m+n} S_i \leqslant t$.

Hence (1) is the same as

$$P\{\max_{1 \leqslant i \leqslant m+n} S_i \leqslant t, \ R_{m,n} = \rho\}; \quad t \geqslant m-n, \quad 2 \leqslant \rho \leqslant 2n+1 \tag{2}$$

2. Joint Distribution of $D^+_{m,n}$ and $R_{m,n}$

Theorem 1. For $t \geqslant m-n \geqslant 0$, $1 \leqslant r \leqslant n$

$$\binom{m+n}{m} P\{D^+_{m,n} \leqslant t, R_{m,n} = 2r\} = 2\binom{m-1}{r-1}\binom{n-1}{r-1} - $$
$$-\binom{m-t-1}{r-1}\binom{n+t-1}{r-1} - \binom{m-t-1}{r}\binom{n+t-1}{r-2} \tag{3a}$$

and for $t \geqslant m-n \geqslant 0$, $1 \leqslant r \leqslant n-1$,

$$\binom{m+n}{m} P\{D^+_{m,n} \leqslant t,\ R_{m,n} = 2r+1\} = \binom{m-1}{r}\binom{n-1}{r-1} +$$

$$+\binom{m-1}{r-1}\binom{n-1}{r} - 2\binom{m-t-1}{r}\binom{n+t-1}{r-1} \qquad (3b)$$

Proof. PART I. First let $R_{m,n} = 2r$

The total number of runs in a path being $2r$, there are r runs* each of $(+1)$'s and (-1)'s and the number of admissible paths is

$$2\binom{m-1}{r-1}\binom{n-1}{r-1} \qquad (4)$$

the factor 2 accounting for the two possibilities that the first run is one of $(+1)$'s or of (-1)'s. From this we have to subtract the number of paths which violate the condition $\max S_i \leqslant t$. Now two cases arise:

Case 1. The path with $\max S_i > t$ starts with a positive run and ends with a negative one.

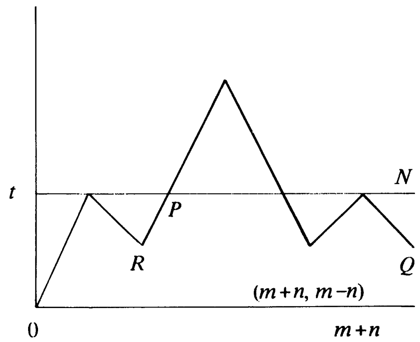

Fig. 1a

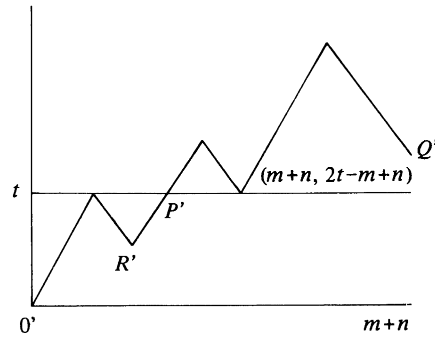

Fig. 1b

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* We shall find it convenient to call a run of $(+1)$'s a positive run and a run of (-1)'s a negative run, and to abreviate the phrase 'Number of positive (negative) runs in a path OP' by $\rho_p(OP)$ and $\rho_n(OP)$.

For a path OQ belonging to this category, let P be the first crossing point with the line $y = t$. Applying the operation* α on OP and φ on PQ, we get the path $O'Q'$ (Fig. 1b). Symbolically,

$$O'Q' = \alpha(OP) + \varphi(PQ)$$

Now the characteristic of the segment PQ is that it starts with a step $+1$ and ends with a -1. Hence the φ –operation applied on PQ yields the segment $P'Q'$, again starting with a $+1$ and ending with a -1. Also $\rho_p(PQ) = \rho_n(PQ)$ and the positive runs in PQ correspond to negative runs in $P'Q'$ and viceversa; hence runs of the two kinds are equal in number in $P'Q'$ as well, with the first run in $P'Q'$ (corresponding to the last one in PQ) becoming a positive run. Also the last runs RP in OP and $R'P'$ in $O'P'$ are positive runs as each one leads to the line $y = t$ from below. Thus both in PQ and $P'Q'$ the first run is in continuation of a positive run and

$$\rho_p(O'Q') = \rho_p(OQ) = \rho_n(O'Q') = \rho_n(OQ)$$

Q is the point $(m+n, m-n)$ and the β- operation on PQ (viz. reflection in $PN : y = t$) transfers Q to Q' $(m+n, 2t-m+n)$. A further γ- operation on it leaves the end points undisplaced. Hence the $\beta * \gamma \equiv \varphi$- operation on PQ while retaining P' as an end point brings the other end point to the position $Q'(m+n, 2t-m+n)$, so that there result $(n+t)$ positive and $(m-t)$ negative steps in $O'Q'$. Thus each one of the paths violating the condition max $S_i \leqslant t$ corresponds to one having $(n+t)$ positive and $(m-t)$ negative steps as also r positive and r negative runs. Hence** the total number of paths of this type is

$$\binom{n+t-1}{r-1}\binom{m-t-1}{r-1} \tag{5}$$

* The operations α, β, γ have been defined by Kanwar Sen [1]. The φ- operation (equivalent to $\beta * \gamma$) is a new one, transforming the path $(\theta_1, \theta_2, ..., \theta_n)$ into the path $(-\theta_n, -\theta_{n-1}, ..., -\theta_1)$.

** The 1-1 correspondence between the sets of paths here and et seq can be easily demonstrated.

Case 2. The path with max $S_i > t$ starts with a negative run and ends with a positive one.

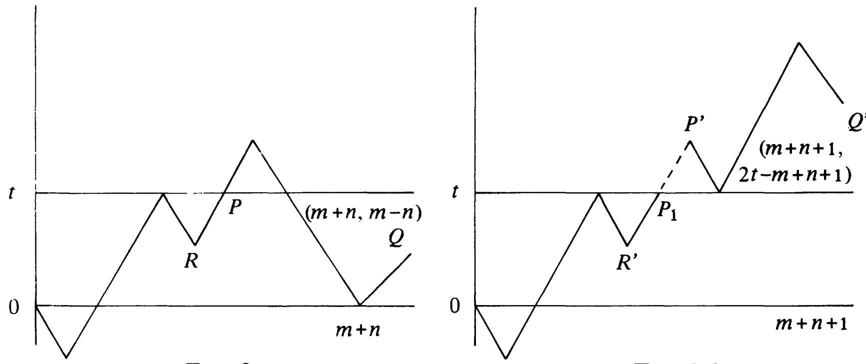

Fig. 2 a *Fig. 2 b*

For such a path OQ (Fig. 2a), let P be the first crossing point with the line $y = t$. Since the last run in PQ is a positive one, the φ - operation transforms it into a segment beginning with a negative run at P; consequently P loses its identity as the first crossing point and a 1-1 correspondence cannot be set up. However, if we adjoin a unit positive segment P_1P' at P_1 (Fig. 2b) and join $\varphi(PQ)$ at P', the identity of P_1 as the point of first crossing is maintained and the unit segment P_1P' is followed by a negative run corresponding to the last positive run in OQ. The segment OR, R being the initial point of the run passing through P, starts as well as ends with a negative run. Hence

$$\rho_n(OR) = \rho_p(OR) + 1 = s, \text{ say}$$

Now

$$\rho_p(OQ) = \rho_n(OQ) = r$$

hence

$$\rho_p(RQ) = r - s + 1 = \rho_n(RQ) + 1$$

Further $O'R'$ is the same as OR, $R'P'$ consists of one positive run and for $P'Q' \equiv \varphi(PQ)$.

$$\rho_p(P'Q') = \rho_n(RQ) \text{ and } \rho_n(P'Q') = \rho_p(RQ)$$

hence

$$\rho_p(O'Q') = r = \rho_n(O'Q') - 1$$

The number of runs is exhibited in the first three columns of the table below:

Table 1

Segment	$R_{m,n} = 2r$		$R_{m,n} = 2r + 1$	
	Positive runs	Negative runs	Positive runs	Negative runs
OQ	r	r	$r+1$	r
$OR \equiv O'R'$	$s - 1$	s	s	s
RQ	$r - s + 1$	$r - s$	$r - s + 1$	$r - s$
$R'P'$	1	0	1	0
$P'Q' \equiv \varphi(PQ)$	$r - s$	$r - s + 1$	$r - s$	$r - s + 1$
$O'Q'$	r	$r + 1$	$r + 1$	$r + 1$

This case differs from Case 1 in the addition of a unit positive segment P_1P'. Hence the terminal point is $Q'(m + n + 1, 2t - m + n + 1)$ and $O'Q'$ comprises $n + t + 1$ positive and $m - t$ negative steps and

$$\rho_p(O'Q') = r = \rho_n(O'Q') - 1$$

consequently the total number of such paths is

$$\binom{n+t}{r-1}\binom{m-t-1}{r} \tag{6}$$

But a path like $O'Q'$ has the segment P_1P' only of unit length; hence from (6) we have to subtract the number of paths which lead from P_1 to Q' beginning with two or more positive steps. On taking away the first unit segment P_1P' from such a path, we are left with a similar path beginning with a positive run of one or more steps and their number, obtained from (6) on replacing n by $n-1$, is

$$\binom{n+t-1}{r-1}\binom{m-t-1}{r} \qquad (7)$$

Thus the required number of paths violating the condition max $S_i \leqslant t$, starting with a negative run and ending with a positive run, is

$$\left[\binom{n+t}{r-1}-\binom{n+t-1}{r-1}\right]\binom{m-t-1}{r}=\binom{n+t-1}{r-2}\binom{m-t-1}{r} \qquad (8)$$

Combining the results (2), (4), (5) and (8) and noting that with m positive and n negative steps the total number of possible paths is $\binom{m+n}{m}$, we prove (3a), generalizing the result obtained by Sujan [3] for the case $m=n$.

PART II. Let $R_{m,\,n}=2r+1$

Now each path has $r+1$ runs of one type and r runs of the other, hence the number of such paths is

$$\binom{m-1}{r}\binom{n-1}{r-1}+\binom{m-1}{r-1}\binom{n-1}{r} \qquad (9)$$

From this we have to subtract the number of paths which violate the condition max $S_i \leqslant t$. Now two cases arise:

Case 3. A path with max $S_i > t$ has $r+1$ positive and r negative runs.

The numbers of positive and negative runs in the various segments after the interposition of a unit positive segment P_1P' (fig. 3a) are

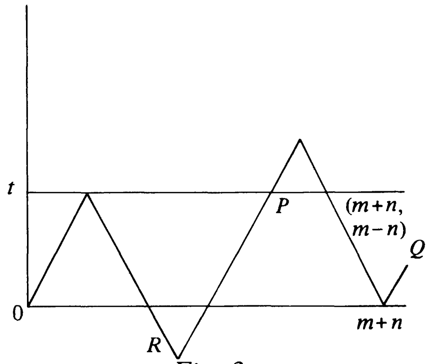

Fig. 3a

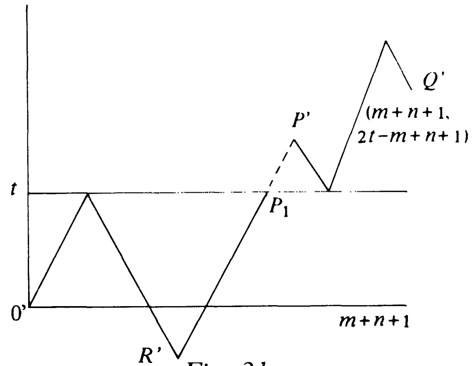

Fig. 3b

exhibited in the last two columns of Table 1. The terminal point is $Q'(m + n + 1, 2t - m + n + 1)$ and $O'Q'$ consists of $n + t + 1$ positive and $m - t$ negative steps; also $\rho_n(O'Q') = \rho_p(O'Q') = r + 1$. Consequently as in Case 2 the number of possible paths now is

$$\binom{n+t}{r}\binom{m-t-1}{r} - \binom{n+t-1}{r}\binom{m-t-1}{r} =$$

$$= \binom{n+t-1}{r-1}\binom{m-t-1}{r} \qquad (10)$$

Case 4. A path with max $S_i > t$ has r positive and $r + 1$ negative runs which is one more than in Case 1.

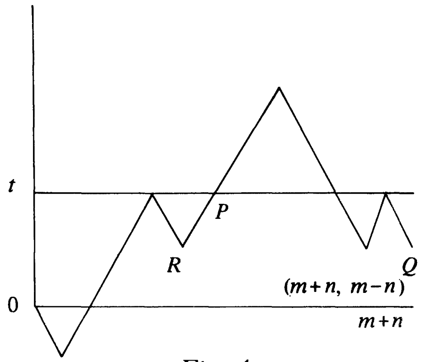

Fig. 4a

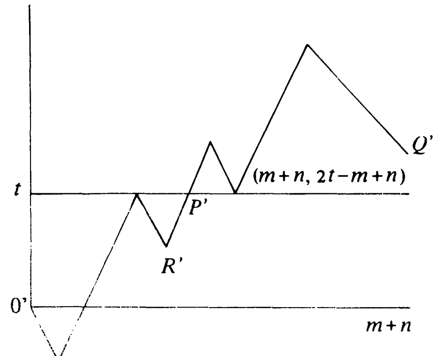

Fig. 4b

Proceeding as in Case 1, each path violating the condition max

$S_i \leqslant t$ is seen to contain $n+t$ positive and $m-t$ negative steps, as also r positive and $r+1$ negative runs. Hence their number is

$$\binom{n+t-1}{r-1}\binom{m-t-1}{r} \tag{11}$$

Combining the results (9), (10) and (11), we prove (3b), generalizing Sujan's another result.

3. Limiting Distribution

Theorem 2. If g, u, y be some fixed finite constants, $m=N+g\sqrt{N}+0(1)$ and $n=N-g\sqrt{N}+0(1)$, then*

$$\lim_{N\to\infty} P\{D^+_{m,n} \leqslant (g-y)\sqrt{N},\ R_{m,n} \leqslant N+u\sqrt{N}\} =$$

$$=\frac{1}{\sqrt{\pi}}(1-e^{g^2-y^2})\int_{-\infty}^{u} e^{-x^2}\,dx \tag{11a}$$

Proof. If a, g, h, k, w, x denote finite constants, N denotes a large number and $0<x<1$, then using Stirling's asymptotic formula:

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* To show that $g^2 \leqslant y^2$, we see that the terminal point of the path is $(m+n, m-n)$. Hence

$$\text{max. ordinate of the path} \equiv D^+_{m,n} \geqslant m-n \geqslant 0$$

by hypothesis of Th. 1. Thus the condition $D^+_{m,n} \leqslant (g-y)N^{1/2}$ is meaningful for $g-y \geqslant 0$. Also from (11a) and (1),

$$(g-y)N^{1/2}=t \text{ has to be} \geqslant m-n=2gN^{1/2}+0(1),$$

by hypothesis of Th. 2,

i.e. $$g-y \geqslant 2g, \quad\text{or}\quad g+y \leqslant 0$$

Also $g-y \geqslant 0$, as shown above; hence the result.

$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n} [1 + 0(1/n)]$$

it is found that for large N,

$$(aN + wN^x + g)! / [\sqrt{2\pi}\, (aN)^{aN + wN^x + g + 1/2}] =$$

$$= \exp [-(aN + wN^x + g) +$$

$$+ (aN + wN^x + g + \frac{1}{2}) \log\{1 + (wN^x + g)/(aN)\}] [1 + 0(N^{-1/2})]$$

$$= \exp\left[-aN + \left(wN^x + g + \frac{1}{2}\right)\left(\frac{wN^x + g}{aN}\right) + \left(aN + wN^x + g + \frac{1}{2}\right) \times\right.$$

$$\left.\times \left\{-\frac{w^2N^{2x} + 2wgN^x}{2a^2N^2} + \frac{w^3N^{3x}}{3a^3N^3} + 0\left(\frac{1}{N^2}\right) + 0\left(\frac{1}{N^{3-2x}}\right)\right\}\right] \times$$

$$\times \left[1 + 0\left(\frac{1}{\sqrt{N}}\right)\right] =$$

$$= \exp [-aN + (w^2/2a)N^{2x-1} + 0(N^{x-1}) + 0(N^{3x-2}) + 0(N^{-1/2})]$$

$$= \exp [-aN + (w^2/2a)N^{2x-1}] [1 + 0(1) + 0(N^{2x-1})]$$

whence

$$\binom{N + g\sqrt{N} + k}{\frac{1}{2}N + wN^x + h} = \frac{2^{N + g\sqrt{N} + k + 1/2}}{\sqrt{\pi N}} \times$$

$$\times \exp\left[-\frac{1}{2}g^2 - 2w^2N^{2x-1} + 2wgN^{x-1/2}\right] \times$$

$$\times \{1 + 0(1) + 0(N^{2x-1})\}$$

When $x > 1/2$, this as $N \to \infty$ diminishes like $\exp(-2w^2N^{2x-1})$. When $x \leqslant 1/2$, the exponent [ ] is $0(1)$ and the number $0(N^x)$ of combinatorial expressions which are of the same order of magnitude is largest when $x = 1/2$. Hence we take $x = 1/2$.

For large N and finite a, f, k, $(a > 0)$

$$(aN + f\sqrt{N} + k)! / [\sqrt{2\pi}\, (aN)^{aN + f\sqrt{N} + k + 1/2}] =$$

$$= \exp\left[-(a N + f\sqrt{N} + k) + \quad a N + f\sqrt{N} + k + \frac{1}{2}\right] \times$$

$$\times \left\{\frac{f\sqrt{N} + k}{a N} - \frac{f^2 N + 2 f k \sqrt{N}}{2 a^2 N^2} + \frac{f^3}{3 a^3 N^{3/2}} + 0\left(\frac{1}{N^2}\right)\right\}\Bigg] =$$

$$= \exp\left[-a N + \frac{f^2}{2 a} + \frac{f}{a\sqrt{N}}\left(k + \frac{1}{2} - \frac{f^2}{6 a}\right) + 0\left(\frac{1}{N}\right)\right]$$

We thus have for ready reference the following asymptotic behaviour:

For large N and finite g, h, k, w

$$\binom{N + g\sqrt{N} + k}{\frac{1}{2} N + w\sqrt{N} + h} \Big/ [2^{N + g\sqrt{N} + k}\sqrt{2/\pi N}\,] =$$

$$= \exp\left[\frac{1}{2} g^2 - w^2 - (g - w)^2 + \frac{1}{\sqrt{N}}\left\{g k + \frac{1}{2} g - \frac{1}{6} g^3 - \right.\right.$$

$$\left. - 2 w h - w + \frac{2}{3} w^3 - 2 (g - w)\left(k - h + \frac{1}{2}\right) + \frac{2}{3} (g - w)^3\right\} +$$

$$\left. + 0\left(\frac{1}{N}\right)\right] = \exp - \frac{1}{2}\,[(g - 2 w)^2 - N^{-1/2}\,\{2 (g - 2 w) (2 h - k) -$$

$$- g + g (g - 2 w)^2\}]\ \{1 + 0\,(1/N)\}$$

We now investigate the asymptotic behaviour of

$$P_1 = \left[2\binom{m-1}{r-1}\binom{n-1}{r-1} - \binom{m-t-1}{r-1}\binom{n+t-1}{r-1} - \right.$$

$$\left. - \binom{m-t-1}{r}\binom{n+t-1}{r-2}\right] \div \binom{m+n}{m}$$

when, with $m-1=N+g\sqrt{N}$, $n-1=N-g\sqrt{N}$, $t=(g-y)\sqrt{N}$ and $r=0(N)$, g, y being finite, N tends to ∞.

We observe that for large N, combinatorial expressions above are dominant when r is about half of m or n or $m-t$ or $n+t$. We, therefore, take $r-1=\frac{1}{2}N+w\sqrt{N}$. Then

$$P_1 \sim \left[2\binom{N+g\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}}\binom{N-g\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}}-\right.$$

$$-\binom{N+y\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}}\binom{N-y\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}}-$$

$$\left.-\binom{N+y\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}+1}\binom{N-y\sqrt{N}}{\frac{1}{2}N+w\sqrt{N}-1}\right]\div\binom{2N+2}{N+g\sqrt{N}+1} \qquad (12)$$

In magnitude the 2nd product in $(12)=2^{N+y\sqrt{N}}\sqrt{2/\pi N}\times$ $\times\exp\left[-\frac{1}{2}(y-2w)^2+\frac{1}{2\sqrt{N}}y\{(y-2w)^2-1\}\right]\times$ [a similar expression with y replaces by $-y$] $[1+0(1/N)]$

$$=\frac{2^{2N+1}}{\pi N}\exp\left[-y^2-4w^2-\frac{4y^2w}{\sqrt{N}}\right][1+0(1/N)]$$

$$=\frac{2^{2N+1}}{\pi N}e^{-y^2-4w^2}\left[1-\frac{4y^2w}{\sqrt{N}}+0(1/N)\right] \qquad (13)$$

In magnitude the 3rd product in (12) after some simplification

$$=[\text{Expression }(12)]\times\exp\frac{4y}{\sqrt{N}}$$

$$=\frac{2^{2N+1}}{\pi N}e^{-y^2-4w^2}\left[1+\frac{4y(1-wy)}{\sqrt{N}}+0(1/N)\right]$$

The 1st product in (12) = 2 [Expression (13) with y replaced by g].

The divisor in (12) = $2^{2N+2}\sqrt{1/\pi N}\, e^{-g^2} + 0\,(N^{-3/2})$

Hence $P_1 = \dfrac{1}{2\sqrt{2\pi N}}\, e^{-4w^2}\left[2\left(1 - \dfrac{4g^2 w}{\sqrt{N}}\right) - e^{g^2 - y^2} \times\right.$

$\left.\times\left\{2 + \dfrac{4y\,(1 - 2wy)}{\sqrt{N}}\right\} + 0\,(1/N)\right]$

$= \dfrac{1}{\sqrt{\pi N}}\, e^{-4w^2}\,[1 - e^{g^2 - y^2} - \dfrac{2}{\sqrt{N}}\,\{2g^2 w +$

$+ e^{g^2 - y^2}\,(y - 2y^2 w)\} + 0\,(1/N)]$

Again,

$$P_2 = \left[\binom{m-1}{r}\binom{n-1}{r-1} + \binom{m-1}{r-1}\binom{n-1}{r} - \right.$$

$$\left. - 2\binom{m-t-1}{r}\binom{n+t-1}{r-1}\right] \div \binom{m+n}{m}$$

$$\equiv \left[\binom{N + g\sqrt{N}}{\frac{1}{2}N + w\sqrt{N} + 1}\binom{N - g\sqrt{N}}{\frac{1}{2}N + w\sqrt{N}} + \right.$$

$$+ \binom{N + g\sqrt{N}}{\frac{1}{2}N + w\sqrt{N}}\binom{N - g\sqrt{N}}{\frac{1}{2}N + w\sqrt{N} + 1} -$$

$$\left. - 2\binom{N + y\sqrt{N}}{\frac{1}{2}N + w\sqrt{N} + 1}\binom{N - y\sqrt{N}}{\frac{1}{2}N + w\sqrt{N}}\right] \Big/ \binom{2N+2}{N + g\sqrt{N} + 1}$$

$= \dfrac{1}{2\sqrt{\pi N}}\, e^{-4w^2}\,[e^{2(g - 2w)/\sqrt{N}} + e^{-2(g + 2w)/\sqrt{N}} -$

$$- 2\, e^{g^2 - y^2 + 2(y - 2w)/\sqrt{N}}] = \frac{1}{\sqrt{\pi N}} e^{-4w^2} [1 - e^{g^2 - y^2} -$$

$$- \frac{2}{\sqrt{N}} \{2w + e^{g^2 - y^2} (y - 2w)\} + 0\,(1/N)]$$

Putting $w \equiv (r - 1) N^{-\frac{1}{2}} - \frac{1}{2}\sqrt{N}$, the highest order terms $0\,(N^{-\frac{1}{2}})$ give, for g, u, y some fixed finite constants,

$$P_N \equiv P\, \{D^+_{m,n} \leqslant (g - y)\sqrt{N},\ R_{m,n} \leqslant N + u\sqrt{N}\}$$

$$= 2 \sum_{r=1}^{\frac{1}{2}N + \frac{1}{2}u\sqrt{N}} \frac{1}{\sqrt{\pi N}} (1 - e^{g^2 - y^2})\, e^{-4w^2}$$

Since an increment of unity in r corresponds to an increment $1/\sqrt{N} = dw$ in w,

$$\lim_{N \to \infty} P_N = \frac{2}{\sqrt{\pi}} (1 - e^{g^2 - y^2}) \int_{-\infty}^{\frac{1}{2}u} e^{-4w^2}\, dw$$

$$= \frac{1}{\sqrt{\pi}} (1 - e^{g^2 - y^2}) \int_{-\infty}^{u} e^{-x^2}\, dx \qquad (14)$$

The contribution from terms $0\,(1/N)$ is

$$\frac{2}{N\sqrt{\pi}} [\sum_{r=1}^{\frac{1}{2}N + \frac{1}{2}u\sqrt{N}} e^{-4w^2} \{2\,(y^2 w - y + w)\, e^{g^2 - y^2} - 2w\,(g^2 + 1)\}] \sim$$

$$\sim \frac{2}{\sqrt{\pi N}} \int_{-\infty}^{u} e^{-x^2} [(y^2 x - 2y + x)\, e^{g^2 - y^2} - x\,(1 + g^2)]\, dx$$

Since the integral is convergent, this tends to zero as $N \to \infty$. Similarly for terms of order smaller than $0\,(1/N)$. In fact, the integrand would always be e^{-x^2} multiplied by a polynomial in x, and the integral would, therefore, be convergent; and this multiplied by a negative power of N would tend to zero. (14) shows that $R_{m,n}$ is asymptotically a normal variate not depending upon g, i.e. upon $m - n$; also $D^+_{m,n}$ and $R_{m,n}$ are asymptotically mutually independent.

ACKNOWLEDGEMENTS

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