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ON THE OUTSTANDING ELEMENTS AND RECORD VALUES IN THE EXPONENTIAL AND GAMMA POPULATIONS

G. S. Lingappaiah
Department of Mathematics
Sir George Williams Campus
Concordia University. Montreal, Canada

SUMMARY

Outstanding elements and record values are discussed in this paper as related to exponential and gamma populations. First, the problem of prediction is considered when there are available, k sets of independent observations from a general-type exponential distribution. In such a case, prediction of the n_k-th record value in the k-th set is made in terms of n_i-th ($i = 1, ..., k - 1$) record values from other ($k - 1$) sets. For this purpose a predictive distribution is obtained. Secondly, the distribution of the sum of record values as well as that of a linear combination of record values are obtained for the exponential case. Probability integrals of the sum of record values and the probability integral of the sum of outstanding elements are suggested for all values. Then, the distribution of the n-th record values in a gamma population is put in a closed form. Further, the distribution of the linear combination of the spacings of outstanding elements as well as that of the linear combination of outstanding elements themselves are obtained. Finally the distribution of a ratio of two record values is obtained.

Key words: Record values; Outstanding elements; Prediction; linear combinations; Bayesian inference.

1. Introduction

Discussion on record values and the outstanding elements has received much attention recently. Chandler [3] discusses the distribution of the lower records as well as the distribution of the difference of the orders of record values. Foster and Stuart [7] consider distribution free tests for the randomness of a series of observations based on the linear function of upper and lower records. Tata [20] gives the limit theorems and also a characterization of the exponential distribution in terms of record values. Nagaraja [19] gives the condition for the existence of the expected value of the n-th record value and also gives the bound for the expected value. Nagaraja [18] gives further characterization of uniqueness of $F(x)$ based on the conditional expectation of the record values.

Nagabhusanam *et al.* [17] discuss the outstanding values as the upper record values of the sequence $\{\log[1 - F(x)] - \log[1 - F(x_i)]\}$, $1 \leqslant i < \infty$ and also give the position of the mean of the r outstanding values. Another aspect dealt with in this paper is the problem of prediction, which is analysed by many authors as can be seen in the references. For example Lawless [11], Lingappaiah [14, 15, 16], Faulkenberry [5], Kaminsky [9], Fertig and Mann [6], Hahn [8], all treat this problem of prediction from the classical point of view using the distribution of the order statistics, while Dunsmore [4], Ling [13], Bancroft and Dunsmore [2] and Aitcheson and Dunsmore [1] deal with the same problem from the Bayesian point of view. What is being done in this paper, firstly is to obtain a predictive distribution using Bayesian approach and from this predictive distribution, the n_k-th record value in the k-th set is predicted in terms of the record values in other $(k - 1)$ sets and the procedure to get a prediction interval is suggested along with the variance of the n_k-th record value. Next, the distribution of the sum of n record values is obtained for the exponential case. When the underlying distribution is gamma, the distribution of the n-th record value is put in the closed form. Also, for the exponential case, distribution of the ratio of two record values is given. Next, the linear combination of record values, linear combination of outstanding values and also a linear combination of the spacings of the outstanding values are considered and the distributions of all these linear combinations are obtained. Probability integral for the sum of record values as well as that of the sum of outstanding values are suggested for all values. Also, the probability function of the ratio of two record values is also given.

2. Preliminaries

x_{u_n} is the n-th record value of a sequence of independent and identically distributed random variables if

$$u_n = \min \{j \mid j > u_{n-1}, x_j > x_{u_{n-1}}\} \tag{1}$$

with $u_0 = 1$ and t_j is an outstanding element where

$$t_j = \log \{[1 - F(x)]/[1 - F(x_{u_j})]\} \tag{2}$$

that is, t_j is the j-th upper record value of the sequence

$$\log \{[1 - F(x)]/[1 - F(x_i)]\} \quad 1 \leqslant i < \infty \tag{3}$$

Suppose we have a distribution such that its distribution function $F(x)$ is of the form [general-exponential type]

$$F(x) = 1 - \exp [-\{h(\theta, x)\}] \tag{4}$$

where $h(\theta, x)$ is differentiable both w.r.t. θ and x and further $x > 0$, $\theta > 0$. Then form (2) and (3), we have

$$t_j = h(\theta, x_{u_j}) - h(\theta, x) \tag{4a}$$

In the case of the exponential distribution,

$$f(x) = \theta \exp(-\theta x), \quad x > 0, \theta > 0 \tag{4b}$$

equation (2) reduces to,

$$t_j = \theta\, (y_j - x) \tag{5}$$

where $y_j = x_{u_j}$ and $x = x_1$ with u_j as given above in (1). From Nagaraja [19], we have, for $n \geqslant 1$,

$$\begin{aligned} p\,[y_n > x \mid y_{n-1} = y] &= [1 - F(x)]/[1 - F(y)], \quad x > y \\ &= 1, \qquad\qquad\qquad\qquad\quad x \leqslant y \end{aligned} \tag{6}$$

and from this it follows that,

$$1 - F_{u_n}(x) = \{1 - F(x)\} \sum_{i=0}^{n} \{-\log[1 - F(x)]\}^i / i! \qquad (7)$$

From (4) and (7), we get

$$1 - F_{u_n}(x) = [\exp\{-h(\theta, x)\}] \sum_{i=0}^{n} \{h(\theta, x)\}^i / i! \qquad (7a)$$

differentiating (7a), we get

$$f_{u_n}(x) = f(y_n) = \{\exp[-h(\theta, x)]\} [h(\theta, x)]^n \, h'_x(\theta, x)/n! \qquad (8)$$

where h'_x is the derivative of $h(x, \theta)$ w.r.t. to x.

For the exponential case, (8) is

$$f(y_n) = \{\theta \, (\theta \, y_n)^n \exp[-(\theta \, y_n)]\}/n! \qquad (9)$$

Now, Nagabhushanam *et al.* [17], consider the joint distribution of the upper record values given by (3) and also the upper record values of the sequence $\{\log F(x) - \log F(x_i)\}$, $1 \leq i < \infty$, and the first observation x. From this joint distribution, the distribution of t_i's is obtained as,

$$f(t_1, t_2, \ldots t_s) = e^{-t_s}, \quad 0 < t_1 < t_2 < \ldots < t_s < \infty \qquad (10)$$

and further x and t's are independent.

Using (2), we get

$$e^{-t_i} = [1 - F(y_i)]/[1 - F(x)] \qquad (10a)$$

and hence

$$e^{-t_s} f(x) \, (dt_1 \ldots dt_s \, dx) = \left[\prod_{i=1}^{s-1} \frac{f(y_i) \, dy_i}{1 - F(y_i)}\right] \left[\frac{f(y_s) f(x) \, dy_s \, dx}{1 - F(x)}\right] \qquad (10b)$$

From (10) and (10b), we get

$$f(y_1, ..., y_s) = [-\log\{1 - F(y_1)\}] \left[\prod_{i=1}^{s-1} \frac{f(y_i)}{1 - F(y_i)} f(y_s)\right] \qquad (10c)$$

$$0 < y_1 < y_2 < \cdots < y_s < \infty$$

For the exponential case, (10c) reduces to

$$f(y_1, ..., y_s) = \theta^{s+1}\, y_1\, e^{-y_s}, \quad 0 < y_1 < y_2 < \cdots < y_s < \infty \qquad (11)$$

and (9) can also be obtained from (11) by integrating out $y_1, ... y_{s-1}$, when (10c) gives

$$f_s(y) = [-\log\{1 - F(y)\}]^s f(y)/s! \qquad (11a)$$

where $f_s(y)$ denotes the p df of y_s.

3. Predictive distribution

Let there be k sets of independent observations from the general-type exponential distribution given by (4). Let $z_i = y_{n_i}$, $i = 1, ... k$, be the n_i-th record value for the set i. Then the distribution of the n_1-th record value is given by (8) as

$$f(z_1 | \theta) = \exp[-h(\theta, z_1)]\, [h(\theta, z_1)]^{n_1}\, h'(\theta, z_1)/n_1! \qquad (13)$$

Taking the prior for θ as $g(\theta) = e^{-\theta}$, $\theta > 0$, we have $f(\theta | z_1)$. Now take the second set of observations and consider the distribution of the n_2-th record value z_2, that is $f(z_2 | \theta)$ which is similar to (13). Now taking $f(\theta | z_1)$ as the prior for the second stage (second set) and along with $f(z_2 | \theta)$, we get the predictive distribution at stage 2 as $f(z_2 | z_1)$. Continuing on this line and taking the posteriori at stage $(k - 1)$, that is $f(\theta | z_1, ..., z_{k-1})$ as the prior for stage k along with the distribution of the n_k-th record value z_k, which is $f(z_k | \theta)$, similar to (13), we get the predictive distribution for z_k at $f(z_k | z_1, ..., z_{k-1})$.

For example, for the exponential distribution given by (4b), we have, the distribution of z_1 as

$$f(z_1 | \theta) = e^{-\theta z_1} z_1^{n_1} \theta^{n_1+1} / n_1! \qquad (14)$$

Now with the above prior $g(\theta)$, we get from (14),

$$f(\theta | z_1) = e^{-\theta(1+z_1)} \theta^{n_1+1} (1 + z_1)^{n_1+2} / \Gamma(n_1 + 2) \qquad (15)$$

Further, integrating out θ from $f(z_2 | \theta)$ $f(\theta | z_1)$, we get the predictive distribution at stage 2 as

$$f(z_2 | z_1) = (1 + S_1)^{n_1+2} z_2^{n_2} / (1 + S_2)^{N_2+3} B(n_2 + 1, n_1 + 2) \qquad (16)$$

where $N_i = n_1 + \cdots + n_i$, $S_i = z_1 + \cdots + z_i$, $i = 1, 2, ..., k$.

Similarly, from $f(\theta | z_1, ..., z_{k-1})$ and the distribution of z_k, n_k-th record value, that is $f(z_k | \theta)$, similar to (14), we get the predictive distribution of z_k as

$$f(z_k | z_1, ..., z_{k-1}) = \frac{(1 + S_{k-1})^{N_{k-1}^{+k}} z_k^{nk}}{B(n_k + 1, N_{k-1} + k)(1 + S_k)^{N_k+k+1}} \qquad (17)$$

From (17), we can get the prediction interval for z_k as $P(z_k < a) =$ $= 1 - \beta$ for set β, Procedure to operate (17) is as follows. Consider k independent set of observations from (4b) and take n_i-th record value $i = 1, 2, ..., k - 1$ from first $(k - 1)$ sets and put them in (17) which turns out to be a function of z_k (that is n_k-th record value) only and then obtain the prediction interval $P(z_k < a) = 1 - \beta$ by integrating. If $n_1 = \cdots = n_k = n$, (that is taking n-th record value for each of k sets), we get (17) as

$$f(z_k | z_1, ..., z_{k-1}) = \frac{z_k^n (1 + S_{k-1})^{n(k-1)+k}}{B[n + 1, n(k - 1) + k](1 + S_k)^{nk+k+1}} \qquad (18)$$

From (17), we get the r-th moment about origin $\mu'_{r(k)}$ for z_k as

$$\mu'_{r(k)} = (1 + S_{k-1})^r \, B(n_k + r + 1, N_k + k - r) / B(n_k + 1, N_{k-1} + k) \quad (19)$$

from which we have the variance of z_k as

$$\sigma^2_{(k)} = (1 + S_{k-1})^2 \, (n_k + 1)(n_k + N_k + k)/(N_k + k - 1)^2 (N_k + k - 2) \quad (20)$$

if all $n_i = n$, $i = 1, 2, \ldots, k$, then (20) reduces to

$$\sigma^2_{(k)} = (1 + S_{k-1})^2 (n + 1)[n(k + 1) + k]/[k(n + 1) - 1]^2 [k(n + 1) - 2] \quad (20a)$$

4. Distribution of the Sum

Now the characteristic function of $y_1 + \cdots + y_s = y$ using (11), is,

$$\phi_y(t) = 1 \Big/ \left[\left(1 - \frac{it}{\theta}\right)\left(1 - \frac{2it}{\theta}\right) \cdots \left(1 - \frac{(s-1)it}{\theta}\right)\right]\left[1 - \frac{sit}{\theta}\right]^2 \quad (21)$$

and inverting (21) we get the distribution of y as

$$f(y) = \frac{(-1)^{s-1}}{(s-1)!} \sum_{k=1}^{s-1} \theta(-1)^k \binom{s-1}{k-1} \frac{k^{s-1}}{s-k} e^{-\theta y/k} + [s^{s-3}/(s-1)!]\,\theta^2 y \, e^{-\theta y/s}$$
$$\sum_{k=1}^{s-1} \left[\frac{s^{s-2}}{(s-1)!}\right]\left(\frac{k}{s-k}\right) \theta \, e^{-\theta y/s} \quad (22)$$
$$0 < y < \infty$$

Probability integral of y can be evaluated for any value of k and s. In (22), the first sum corresponds to the residues at the poles $t = -i\theta/k$, $k = 1, 2, \ldots, s-1$ while the second term *and* the last sum correspond to the residue at the multiple pole at $t = -i\,\theta/s$. From Nagabhushanam *et al.* [17], we have the distribution of $z = t_1 + \cdots + t_s$ as

$$f(z) = \sum_{k=1}^{s} (-1)^{s-k}\, e^{-z/k}\, k^{s-2}/(k-1)!\,(s-k)! \tag{23}$$

Probability integral of z can easily be evaluated for any value of s and k. From (5), we have

$$\sum_{i=1}^{s} t_i = \theta \sum_{i=1}^{s} (y_i - x) \tag{24}$$

that is,

$$z = \theta\,(y - s\,x) \tag{24a}$$

which gives

$$\theta^2\, \mathrm{Var}\,(y) = \mathrm{Var}\, z + s^2 \tag{25}$$

We can get the Var (y) since we have the Var z from Nagabhushanam *et al.* [17] as

$$\theta^2\, \mathrm{Var}\,(y) = \left[\frac{s\,(s+1)\,(2s+1)}{6} + s^2\right] \tag{26}$$

Of course (26) can also be obtained directly from (22). From (9), we have

$$E(y_k) = \frac{k+1}{\theta}, \quad k = 1, \ldots s, \tag{26a}$$

and from (11), with the joint density of y_k and y_j $(k < j)$, we get

$$E(y_j y_k) = (j+2)(k+1)/\theta^2 \tag{26b}$$

and

$$\mathrm{Cov}(y_j, y_k) = (k+1)/\theta^2 \tag{26c}$$

and finally,

$$\rho(y_j, y_k) = \sqrt{(k+1)/(j+1)} \tag{26d}$$

These (26a) to (26d) can also be obtained by using (5).

5. Gamma population

If $\{x_n\}$ is a sequence of independent random variables each having gamma distribution

$$f(x) = \theta(\theta x)^{\alpha-1} e^{-\theta x}/\Gamma(\alpha) \tag{27}$$

Then from (7), we get, [with $f_n(y) = f_{u_n}(y) = f(y_n)$]

$$f_n(y) = f(y)\,[-\log\{1 - F(y)\}]^n/n! \tag{28}$$

which gives

$$f_n(y) = [f(y)/n!] \sum_{r=0}^{\infty} a_{r,n} F^{n+r} \tag{29}$$

where $a_{r,n}$ is the coefficient of F^r in the expansion of $\left[\sum_{i=0}^{\infty} F^i/(i+1)\right]^n$ and satisfies the relation

$$a_{r,n} = a_{r,n-1} + a_{r-1,n-1}/2 + \cdots + a_{0,n-1}/(r+1) \tag{29a}$$

Noting that in the case of gamma,

$$F(x) = 1 - \sum_{k=0}^{\alpha-1} e^{-\theta x} (\theta x)^k / k! \tag{30}$$

we get (29) as

$$f_n(y) = [1/n!\ \Gamma(\alpha)] \left[\sum_{r=0}^{\infty} a_{r,n} \sum_{j=0}^{n+r} \binom{n+r}{j} (-1)^j \sum_{t=0}^{(\alpha-1)j} b_t(\alpha, j) \cdot \right.$$

$$\left. \cdot\, e^{-\theta y (j+1)}\, \theta\, (\theta y)^{t+\alpha-1} \right] \tag{31}$$

where $b_t(\alpha,j)$ is the coefficient of $(\theta y)^t$ in the expansion of $\left[\sum_{k=0}^{\alpha-1} (\theta y)^k / k!\right]^j$ and b_t's satisfy the relation

$$b_t(\alpha, j) = b_t(\alpha, j-1) + b_{t-1}(\alpha, j-1) + b_{t-2}(\alpha, j-1)/2! +$$

$$+ \cdots + b_{t-\alpha+1}(\alpha, j-1)/(\alpha-1)! \tag{32}$$

If $\alpha = 1$ in (31), last sum on t's vanish and the second sum gives $(1 - e^{-\theta y_n})^{n+r}$ and the first sum on $(1 - e^{-\theta y_n})^{n+r}$ gives $\{-\log[1 - (1 - e^{-\theta y_n})]\}^n$ which is $(\theta y_n)^n$ and so (31) reduces to (9).

6. Linear combinations

6a: Let

$$T = \sum_{i=1}^{s} a_i y_i, \quad a_i > 0 \tag{33}$$

then the characteristic function of T is, using (11),

$$\phi_T(t) = 1 \Big/ \left[\left(1 - \frac{\text{it}\ \Sigma_s}{\theta}\right) \left(1 - \frac{\text{it}\ \Sigma_{s-1}}{\theta}\right) \cdots \right.$$

$$\cdots \left(1-\frac{\text{it}\,\Sigma_2}{\theta}\right)\Bigg]\left[1-\frac{\text{it}\,\Sigma_1}{\theta}\right]^2 \tag{34}$$

where

$$\Sigma_i = a_i + a_{i+1} + \cdots + a_s \tag{34a}$$

and inverting (34), we get the distribution of T as

$$f(T) = \sum_{t=1}^{s-1} \frac{\theta\,(\Sigma_{s-k+1})^{s-1}\, e^{-(\theta T/\Sigma_{s-k+1})}}{\left[\prod\limits_{\substack{j=1\\ j\neq s-k+1}}^{s} (\Sigma_{s-k+1}-\Sigma_j)\right][\Sigma_{s-k+1}-\Sigma_1]} +$$

$$+\,\theta^2\, T\, e^{-(\theta T/\Sigma_1)}\, \Sigma_1^{s-3} \Big/ \prod_{j=2}^{s} (\Sigma_1-\Sigma_j) -$$

$$-\sum_{k=1}^{s-1} \frac{\theta\,(\Sigma_{s-k+1})\,\Sigma_1^{s-2}\, e^{-(\theta T/\Sigma_1)}}{\left[\prod\limits_{i=2}^{s} (\Sigma_1-\Sigma_i)\right][\Sigma_1-\Sigma_{s-k+1}]} \tag{35}$$

If we set $a_1 = a_2 = \ldots = a_s = 1$, then (35) reduces to (22) and a_i's are also such that none of the terms in the denominator of (35) vanishes.

6b: Linear combination of t's:

The characteristic function of $\sum_{i=1}^{s} b_i t_i = w$, $b_i > 0$, is, using (10),

$$\phi_w(t) = 1/(1-\text{it}\,\Sigma_1)(1-\text{it}\,\Sigma_2)\ldots(1-\text{it}\,\Sigma_s) \tag{36}$$

where $\Sigma_i = b_i + b_{i+1} + \cdots + b_s$. Now inverting (36), we get

$$f(w) = \sum_{k=1}^{s} \frac{e^{-(w/\Sigma_{s-k+1})}(\Sigma_{s-k+1})^{s-2}}{\prod\limits_{\substack{j=1\\ j\neq s-k+1}}^{s} (\Sigma_{s-k+1}-\Sigma_j)} \tag{37}$$

If $b_1 = \cdots = b_s = 1$, then (37) reduces to (23) and same condition on b's as a's before.

6c. Linear combination of spacings of t's

Let $\nu_i = t_i - t_{i-1}$, $i = 1, 2, \dots s$ and let

$$\nu = \sum_{i=1}^{s} c_i \nu_i, \quad c_i > 0 \text{ with } t_0 = 0, \quad c_i \neq c_j, \quad i, j = 1, 2, \dots s$$

Then from (10), it follows that the *pdf* of ν_i is

$$f(\nu_i) = e^{-\nu_i}, \quad \nu_i > 0, \quad i = 1, 2, \dots s \tag{38}$$

and further ν_i's are independent. Characteristic function of ν is

$$\phi_\nu(t) = 1 \Big/ \prod_{r=1}^{s} (1 - c_r(\text{it})) \tag{39}$$

Inverting (39), we get the *pdf* of ν as

$$f(\nu) = \sum_{r=1}^{s} c_r^{s-2} \, e^{-\nu/c_r} \Big/ \prod_{\substack{j=1 \\ j \neq r}}^{s} (c_r - c_j) \tag{40}$$

For example, if $s = 4$, and

$$\nu = \nu_1 + 3\,\nu_2 + 4\,\nu_3 + 2\,\nu_4 \tag{40a}$$

we have

$$w = 2\,t_4 + 2\,t_3 - t_2 - 2\,t_1 \tag{40b}$$

and hence $\Sigma_1 = 1$, $\Sigma_2 = 3$, $\Sigma_3 = 4$ and $\Sigma_4 = 2$ in (37). So we can use either (37) or (40) as the case may be. It is also to be noted that

$$\Sigma\, a_i\, v_i = \Sigma\, a_i\, (t_i - t_{i-1}) = \theta\, \Sigma\, a_i\, (y_i - y_{i-1}), \quad i \geqslant 2 \text{ for } i = 1 \tag{41}$$

last term includes y_0. Further, it is easy to observe that

$$\text{Var}\, v = \text{Var} \sum_1^s a_i\, v_i = \sum_1^s a_i^2 \tag{42}$$

and

$$\Sigma\, a_i^2 = \text{Var}\left[\sum_1^s d_i\, t_i\right] \tag{42a}$$

where a_i's and d_i's are related.

So using v_i's, it is easier to calculate the variance of t_i's instead from the distribution of t_i's, since they are not independent. For example, for (40a), we have Var $v = 30$ which is exactly what we get from (40b) using variances and covariances of t_i's.

7. Ratio y_s/y_i, i = 1, 2, ..., s − 1

Now from (11), we get by integrating out $y_{i+1}, ..., y_{s-1}$, the joint density of y_i and y_s as

$$f(y_i, y_s) = y_i^i (y_s - y_i)^{s-i-1}\, e^{-\theta y_s}\, \theta^{s+1}/i!\,(s-i-1)! \tag{43}$$

$$0 < y_i < y_s < \infty$$

Now setting $y_i/y_s = u_i$, we get the *pdf* of u_i by integrating out y_i, as

$$f(u_i) = u_i^i\,(1 - u_i)^{s-i-1}/B\,(i+1, s-i), \quad 0 < u_i < 1 \tag{44}$$

where $B(a, b)$ is the complete beta function, $p\ (y_s > c\ y_i)$ can be found by evaluating $p\ (u_i > u)$ by using incomplete beta function tables.

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