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# A NOTE ON STABILITY OF MULTIVARIATE DISTRIBUTION

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ABSTRACT

*In this note, we give an elementary proof of a characterization for stability of multivariate distributions by considering a functional equation.*

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*Key Words: Characteristic function, stable law, characteristic exponent, functional equation.*

## 1. Introduction.

A probability measure $\mu$ on $\mathbf{R}^d$ is said to be stable if there are sequences $\{X_n\}$, $\{a_n\}$, and $\{b_n\}$ such that $\{X_n\}$ are independent and identically distributed $\mathbf{R}^d$-valued random vectors, $a_n \in \mathbf{R}^d$, $b_n > 0$, and the distribution of $b_n^{-1} \sum_{j=1}^{n} X_j - a_n$ converges to $\mu$. As is well known, $\mu$ is stable if and only if, for each $c_1 > 0$ and $c_2 > 0$, there exist $c > 0$ and $a \in \mathbf{R}^d$ such that the characteristic function $\hat{\mu}$ of $\mu$ satisfies

$$\hat{\mu}(c_1 t)\hat{\mu}(c_2 t) = \hat{\mu}(ct)e^{ia't}, \qquad \forall t \in \mathbf{R}^d. \tag{1}$$

In this case, $c$ is uniquely determined by $c = (c_1^\alpha + c_2^\alpha)^{\frac{1}{\alpha}}$, with the characteristic exponent $0 < \alpha \leq 2$, independent of $c_1$ and $c_2$.

It is natural to ask what if we only assume (1) is satisfied by a single collection of $c_1$, $c_2$, $c$ and $a$. The purpose of this note is to present an elementary proof of the answer: in most of the cases, this much weaker condition characterizes the stability of $\mu$. Our approach is to reduce the characterization problem to solving a functional equation. The main theorem in this note is applied in Zeng (1989) to characterize multivariate stable distributions via random linear statatistics.

## 2. The Main Theorem

The main result of this note is

**Theorem 1.** A probability measure $\mu$ on $\mathbf{R}^d$ is stable if and only if there exist positive numbers $c_1, c_2, c$ and $a \in \mathbf{R}^d$ such that $\frac{c_1}{c}$ and $\frac{c_2}{c}$ are non-commensurable (i.e., there are no integers $m$ and $n$ such that $\left(\frac{c_1}{c}\right)^m = \left(\frac{c_2}{c}\right)^n$), and the equation (1) holds.

We first prove a lemma.

**Lemma 1.** Let $f : (0, \infty) \to \mathbf{R}$ be a monotone function, and let $a, b \in (0, 1)$ be non-commensurable. If

$$f(ax) + f(bx) = f(x), \qquad \forall x > 0, \tag{2}$$

then $f(x) = cx^\alpha$, where $a^\alpha + b^\alpha = 1$, and $c$ is a constant.

*Proof.* Without loss of generality, we can assume that $f$ is nondecreasing, not identically zero, $a < b$, and $a + b = 1$ (i.e., $\alpha = 1$).

Let $c = \inf_{x>0} \left(\frac{f(x)}{x}\right)$, then $c \neq 0$. For any $\delta > 0$, it is clear that if $kx \leq f(x) \leq Kx$ on $(a\delta, \delta)$ for some $k$ and $K$, then the same inequalities hold on $(a\delta, \infty)$. this implies that

$$c = \inf_{0<x\leq\delta} \left(\frac{f(x)}{x}\right), \tag{3}$$

where $\delta$ is any positive number. We will show that for any $\epsilon > 0$, $\frac{f(x)}{x} < c + \epsilon$ for all $x > 0$, so that the lemma follows.

Since $a$ and $b$ are non-commensurable, the set $\{a^m b^n : m, n \in \mathbb{N}\}$ can be indexed as a decreasing sequence $1 = a_0 > a_1 > a_2 > \cdots$, and $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\right) = 1$. Let $\epsilon_1 = \frac{\epsilon}{2|c|}$ (since $c \neq 0$), then there exists $n_0$ such that

$$\frac{a_n}{a_{n+1}} < 1 + \epsilon_1, \qquad \text{for all } n \geq n_0. \tag{4}$$

In light of (3), we can also find $x_0 < \delta$ with

$$\frac{f(x_0)}{x_0} < c + \epsilon_2,$$

where $\epsilon_2 = a a_{n_o}\frac{\epsilon}{2}$. By applying the equation (2) repeatedly, we have

$$f(x_0) = f(a_n x_0) + \sum_{j\in I} f(a_j x_0),$$

where $I \subset \mathbb{N}$ is a finite set, $a_n + \sum_{j\in I} a_j = 1$, and $n \geq n_0$. It follows that

$$f(a_n x_0) = f(x_0) - \sum_{j\in I} f(a_j x_0),$$

and hence

$$f(a_n x_0) < c x_0 + \epsilon_2 x_0 - \sum_{j\in I} c a_j x_0 = c(a_n x_0) + \epsilon_2 x_0, \tag{5}$$

for all $n \geq n_0$. Let $t \in (a a_{n_0} x_0, a_{n_0} x_0]$, then $t \in (a_{n+1} x_0, a_n x_0]$ for some $n \geq n_0$. The monotonicity of $f$ along with (4) and (5) yield that

$$f(t) \leq f(a_n x_0) < c(a_n x_0) + \epsilon_2 x_0 < ct + \epsilon t,$$

i.e.,

$$\frac{f(t)}{t} < c + \epsilon, \qquad \forall t \in (a a_{n_0} x_0, a_{n_0} x_0],$$

and the inequality is valid for all $x > 0$.

**Remark.** If we assume $f$ is nonnegative, Lemma 1 becomes a special case of the Lau-Rao's theorem (1982) on the integrated Cauchy functional equation. We give here a direct simple proof, inspired by Shanbhag (1977), without the nonnegativeness assumption. Following the same lines with slight modification, we can show that the some conclusion holds for a more general functional equation

$$f(x) = \sum_{j=1}^{n} d_j f(a_j x),$$

where the $a_j$'s are not commensurable, and $\sum_{j=1}^{n} d_j a_j^\alpha = 1$.

*Proof of Theorem 1.* Let $\mu$ be a probability measure of $\mathbf{R}^d$, $\hat{\mu}$ its characteristic function. If $\mu$ is stable with characteristic exponent $\alpha$, then for any $c_1 > 0$ and $c_2 > 0$, there exist $c > 0$ and $a \in \mathbf{R}^d$ such that the characteristic function $\hat{\mu}$ of $\mu$ satisfies the equation (1) and $c^\alpha = c_1^\alpha + c_2^\alpha$. This equation on $c$'s can be written as $\left(\frac{c_1}{c}\right)^\alpha + \left(\frac{c_2}{c}\right)^\alpha = 1$. To show the existence of non-commensurable $\frac{c_1}{c}$ and $\frac{c_2}{c}$, we observe the function $\phi(x) = \frac{\log(1-x^\alpha)}{\alpha \log x}$ defined on (0,1). The continuity of $\phi$ on the interval (0,1) implies that for almost all of the pairs $\frac{c_1}{c}$ and $\frac{c_2}{c}$, $\log\left(\frac{c_1}{c}\right) / \log\left(\frac{c_2}{c}\right)$ is irrational, and hence the neccessity follows.

Conversely, let $c_1, c_2, c$ and $a$ be such that

$$\hat{\mu}(c_1 t)\hat{\mu}(c_2 t) = \hat{\mu}(ct)e^{ia't}, \qquad \forall t \in \mathbf{R}^d,$$

and $\beta_1 = \frac{c_1}{c}$ and $\beta_2 = \frac{c_2}{c}$ are non- commensurable. Then $c \geq c_j$, $j = 1, 2$, with the equality only in trivial case, and hence equation (1) can be rewritten as

$$\hat{\mu}(\beta_1 t)\hat{\mu}(\beta_2 t) = \hat{\mu}(t)e^{i\gamma' t}, \qquad \forall t \in \mathbf{R}^d. \tag{6}$$

where $0 < \beta_1, \beta_2 < 1$, excluding the trivial case, and $\gamma \in \mathbf{R}^d$. Notice that the equation (6) also implies that $\mu$ is infinitely divisible, the Lévy canonical representation gives

$$\hat{\mu}(t) = \exp\left\{ iP_1(t) + P_2(t) + \int_{\mathbf{R}^d} \left( e^{iw't} - 1 - \frac{iw't}{1+|w|^2} \right) dV(w) \right\},$$

where $P_1$ and $P_2$ are homogeneous polynomials of degree one and two, respectively, $V$ is a measure on $\mathbf{R}^d$ such that $V(\{0\}) = 0$ and

$$\int_{\mathbf{R}^d} \left( \frac{|w|^2}{1+|w|^2} \right) dV(w) < \infty. \tag{7}$$

Let $\alpha$ be the unique positive number determined by $\beta_1^\alpha + \beta_2^\alpha = 1$. Then the spectral measure $V$ satisfies

$$dV\left(\frac{w}{\beta_1}\right) + dV\left(\frac{w}{\beta_1}\right) = dV(w). \tag{8}$$

Given $B \subseteq \mathbf{R}_+$ and $B \subseteq S^d-1$ (the unit sphere in $\mathbf{R}^d$), let EB denote the set of $w$ such that $w = su$, $s \in E$, $u \in B$. The equation (8) implies that

$$\int_{EB} dV\left(\frac{w}{\beta_1}\right) + \int_{EB} dV\left(\frac{w}{\beta_2}\right) = \int_{EB} dV(w).$$

Given $B \subseteq S^{d-1}$, if we let $N(x,B) = \int_{(x,\infty)B} dV(w)$, then $N(x,B)$ is a monotone decreasing function on $(0,\infty)$, satisfying

$$N\left(\frac{x}{\beta_1}, B\right) + N\left(\frac{x}{\beta_2}, B\right) = N(x,B), \qquad \forall x > 0, \tag{9}$$

Since $\beta_1$ and $\beta_2$ are non-commensurable, it follows from Lemma 1 that $N(x,B) = c(B)x^{-\alpha}$, for any Borel set $B \subseteq S^{d-1}$, and the spectral measure $V$ should be of the form

$$V(EB) = \int_{EB} \frac{ds\,d\Phi(u)}{s^{\alpha+1}}, \qquad \text{for } E \subseteq \mathbf{R}_+, \quad B \subseteq S^{d-1}, \tag{10}$$

where $\Phi$ is a positive finite measure on the unit sphere $S^{d-1}$. Hence, the equation (1) together with (7) and (10) yield that $0 < \alpha \leq 2$, unless $\mu$ is degenerate, and the spectral measure $V$ is zero if $\alpha = 2$, in which case $\mu$ is a normal law, or otherwise, $V$ is given by (10), and the quadratic form $P_2 = 0$, so that $\mu$ is a stable law with characteristic exponent $0 < \alpha < 2$.

3. Further characterizations of stable laws.

The following characterizations of multivariate stable distributions through identically distributed linear statistics (e.g., see Gupta et al) can be easily derived from Theorem 1.

**Theorem 2.** Let $X_1$, $X_2$ and $X_3$ be independent and identically distributed random vectors in $\mathbf{R}^d$. Then $X_1$ has a multivariate stable distribution if and only if there exists $\alpha$, $0 < \alpha \le 2$, and $a_1, a_2 \in \mathbf{R}^d$, such that $2^{1/\alpha}X_1 + a_1$ and $X_1 + X_2$, $3^{1/\alpha}X_1 + a_2$ and $X_1 + X_2 + X_3$ are identically distributed, respectively.

*Proof.* Let $\hat{\mu}$ denote the characteristic function of $X_1$. The given conditions imply that $X_1 + a$, $a \in \mathbf{R}^d$, is identically distributed with $\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} X_1 + \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} X_2$, and we have

$$\hat{\mu}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} t\right)\hat{\mu}\left(\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} t\right) = \hat{\mu}(t)e^{ia't}, \qquad \forall t \in \mathbf{R}^d.$$

Since $\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{3}\right)^{\frac{1}{\alpha}}$ are not commensurable, Theorem 1 implies that $\hat{\mu}$ is stable with characteristic exponent $\alpha$.

We conclude with two special cases when $\alpha = 1$ and $\alpha = 2$.

**Corollary.** Let $X_1, X_2$ and $X_3$ be independent and identically distributed random vectors in $\mathbf{R}^d$. Then

(1). $X_1$ has a multivariate stable distribution with Cauchy marginals if $2X_1$ and $X_1 + X_2$, $3X_1$ and $X_1 + X_2 + X_3$ are identically distributed, respectively.

(2). $X_1$ has a multivariate normal distribution (possibly degenerate) with zero location vector if and only if $\sqrt{2}X_1$ and $X_1+X_2$,$\sqrt{3}X_1$ and $X_1+X_2+X_3$ are identically distributed, respectively.

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