Stochastica XIII-1 (1992), 137-143

ON EXACT CONDITIONALS

ENRIC TRILLAS

1. Introduction.

1.1. Let it be E a Boolean Algebra of propositions $a, b, c, \ldots$ of which those belonging to a previous given subset $T \subset E$ are true and the others in $F = E - T$ are false.

When, in Commonsense Reasoning, it is affirmed a conditional relation

"If a, then b"

(for short $a \Rightarrow b$) it is also affirmed that $a'\ (b+b')+ab = a'+b \in T$, provided that boolean operations + of join, of meet and ' of negation verify the properties: $a \in T$ iff $a' \in F$; $a.b \in T$ iff $a \in T$ and $b \in T$, and $a+b \in T$ iff $a \in T$ or $b \in T$. Then, it is supposed that the relation of Material Conditional associated with T:

$$a \rightarrow_T b \quad \text{iff} \quad a'+b \in T,$$

contains $\Rightarrow : \Rightarrow \subset \rightarrow_T$ [4].

The frequently made hypothesis $\Rightarrow \triangleq \rightarrow_T$ conveys the undesired consequence of $a \Rightarrow b$, if $a \in F$. That fact, important in Formal Reasoning, is not usual in Commonsense Reasoning [1]. It is rare to affirm as a piece of Commonsense Reasoning, something like

"If Madrid is the capital of France, then this is a paper on Logic".

no matter if "this is a paper on Logic" is true or false.

What is actualy supposed satisfies a conditional relation is only the so-called Modus Ponens Rules:

$$\text{If} \quad a \in T \quad \text{and} \quad a \Rightarrow b, \quad \text{then } b \in F,$$

that implies the Modus Tollens Rule:

$$\text{If} \quad b \in F \quad \text{and} \quad a \Rightarrow b, \quad \text{then } a \in F,$$

Definition 1.1.1. Given a set E and a non-empty subset $T \subset E$, a binary relation on E, $\Rightarrow \subset E \times E$, is a T- conditional on E if:

$$a \in T \quad \text{and} \quad a \Rightarrow b \quad \text{implies} \quad b \in T.$$

It is clear that if $\Rightarrow$ is a T-conditional on E and $\approx>$ is another relation on E such that $\approx> \subset \Rightarrow$, then $\approx>$ is a T-conditional on E. Sometimes, when 1.1.1. holds, it is said that T is a t-set (t for true) or a Logic State for the relational structure $(E, \Rightarrow)$.

If $T = \{ \mathbf{1} \}$, as a $\rightarrow_{\mathbf{1}}$ b iff $a' + b, = \mathbf{1}$ iff $a \leq b$, it is clear that the $\{\mathbf{1}\}$-Material Conditional in a Boolean Algebra is its partial order. **1** is the largest element of the Boolean Algebra E.

In what follows we will study such kind of exact relations, T- conditionals (of which T-Material Conditional is the best known) without using any algebraic structure on the ground proposition's set E. We speak of exact as more restrictive than inexact, in the sense of [2] and [4].

1.2 To chain pieces of reasoning it is convenient that a relation $\Rightarrow$ on E, modelizing a conditional, be transitive. But if it is not the case we can extend $\Rightarrow$ to its transitive clausure $\Rightarrow_t$: $a \Rightarrow_t b$ means that $a \Rightarrow a_1$, $a_1 \Rightarrow a_2, ..., a_n \Rightarrow b$, for some propositions $a_1, \ldots, a_n$ in E. It should be realized that $\Rightarrow \subset \Rightarrow_t$.

Theorem 1.2.1. A relation $\Rightarrow$ verifies the Rule of Modus Ponens if and only if $\Rightarrow_t$ does.

Proof. If $\Rightarrow_t$ is a T-conditional, $\Rightarrow$ is a T-conditional. Reciprocally, if $a \in T$ and $a \Rightarrow_t b$, is $a \in T$ and ($a \Rightarrow a_1$, $a_1 \Rightarrow a_2, \ldots, a_n \Rightarrow b$) or $a \in T$, and $a_1 \in T$, and $a_2 \in T, \ldots$, and $a_n \in T$ and $b \in T$.

It is frequently supposed that a T-conditional satisfies the weak condition of reflexivity: $a \Rightarrow a$, for each $a \in E$, translating the usual affirmation "If a, then a". If relation $\Rightarrow$ is not reflexive, it can be extended to its reflexive clausure

$$\Rightarrow_r = \Rightarrow \cup \{(a,a); a \in E\}.$$

Of course $\Rightarrow \subset \Rightarrow_r$.

Theorem 1.2.2. A relation $\Rightarrow$ is a T-conditional if and only if $\Rightarrow_r$ is a T-conditional.

Proof. If $\Rightarrow_r$ is a T-conditional is obvious that $\Rightarrow$ does. Reciprocally, if $a \in T$ and $a \Rightarrow_r b$, it is $a = b$ (and $b \in T$) or $a \neq b$ and then $a \Rightarrow b$ and $b \in T$.

If $\Rightarrow$ is not transitive and reflexive, we can proceed from $\Rightarrow$ to $\Rightarrow_{rt}$:

$$\Rightarrow \subset \Rightarrow_r \subset \Rightarrow_{rt},$$

and $\Rightarrow$ is a T-conditional iff $\Rightarrow_{rt}$ is a T-conditional.

2. T-conditionals.

Next result shows an intrinsec representation of the Material Conditional.

Theorem 2.1. Given (E,T), the relation $\rightarrow_T = (F \times E) \cup (T \times T)$ is the greatest T-conditional.

Proof. Let's consider the set $\mathcal{C}_T = \{\Rightarrow \subset E \times E; \Rightarrow \text{ is a T-conditional}\}$; that set is non-empty, for example $T \times T$ belongs to $\mathcal{C}_T$. Consider

$$\longrightarrow_T = \bigcup_{\Rightarrow \in \mathcal{C}_T} \Rightarrow$$

Such relation is a T-conditional: if $a \rightarrow_T b$, it should be also $a \Rightarrow b$ for some $\Rightarrow \in \mathcal{C}_T$, and then if $a \in T$ it is $b \in T$. Obviously $\longrightarrow_T$ is the greatest T-conditional.

If $a \in T$, for having $a \longrightarrow_T b$ for some $b \in E$, it should be $b \in T$. But if $a \in F$, it is always $a \longrightarrow_T b$ for any $b \in E$, because $\Rightarrow_* = T \times T \cup \{(a, b)\}$ is a T-conditional such that $a \Rightarrow_* b$. Then $\longrightarrow_T = (F \times E) \cup (T \times T)$.

Corollary. A relation $\Rightarrow \subset E \times E$ is a T- conditional if and only if $\Rightarrow \subset \rightarrow_T$.

Proof. By theorem 2.1 if $\Rightarrow$ is a T-conditional, then $\Rightarrow \subset \rightarrow_T$. Reciprocally, if $a \in T$ and $a \Rightarrow b$ it is $a \in T$ and $a \rightarrow_T b$ and, being $\longrightarrow_T$ a T-conditional, $b \in T$.

Theorem 2.3. The T-Material Conditional is a Preorder.

Proof. For $a \in E$, it is $a \in T$ and $a \rightarrow_T a$, or it is $a \in F$ and, as $a \in E$, it is also $a \rightarrow_T a$. Suppose $a \rightarrow_T b$ and $b \rightarrow_T c$. If $a \in F$, as $c \in E$, it is $a \rightarrow_T c$; if $a \in T$, then $b \in T$ and $c \in T$, and $a \rightarrow_T c$.

Corollary. Given a set $A \subset E$, the relation $\rightarrow_A = (E - A) \times E \cup A \times A$ is a preorder, the preorder by A.

If $\Rightarrow$ is a T-conditional such that when $a \in F$ it is $a \Rightarrow b$ for any $b \in E$, then $F \times E \subset \Rightarrow \subset \rightarrow_T$.

If $\{\mathbf{1}\} \subset T$ it is $\rightarrow_{\{\mathbf{1}\}} \subset \rightarrow_T$ and, in that restricted sense of monotonicity, the classical material conditional $\rightarrow_{\{\mathbf{1}\}} = \leq$ is the more conservative: every conditional $a \leq b$ implies the conditional $a \rightarrow_T b$, for any set T containing $\mathbf{1}$.

3. On consequences and conditionals.

Let's consider for any relation $\Rightarrow \subset E \times E$ the mapping $C_{\Rightarrow} : \mathbf{P}(E) - \{\emptyset\} \rightarrow \mathbf{P}(E) - \{\emptyset\}$, given by [3]:

$$\mathbb{C}_{\Rightarrow}(T) = \{x \in E;\ \exists a \in T : a \Rightarrow x\},$$

for each $T \subset E$, $T \neq \emptyset$. It is obvious that $\mathbb{C}_{\Rightarrow}$ is monotone: if $A \subset B$ then $\mathbb{C}_{\Rightarrow}(A) \subset \mathbb{C}_{\Rightarrow}(B)$. It is also obvious that $\Rightarrow_1 \subset \Rightarrow_2$ implies $\mathbb{C}_{\Rightarrow 1}(A) \subset \mathbb{C}_{\Rightarrow 2}(A)$.

Theorem 3.1. Relation $\Rightarrow$ is a T-conditional, for $\emptyset \neq T \subset E$, if and only if $\mathbb{C}_{\Rightarrow}(T) \subset T$.

Proof. If $\mathbb{C}_{\Rightarrow}(T) \subset T$, then if $a \in T$ and $a \Rightarrow b$, as $b \in \mathbb{C}_{\Rightarrow}(T)$, it is $b \in T$, and $\Rightarrow$ is T-conditional. Reciprocally, if $\Rightarrow$ is a T- conditional and $x \in \mathbb{C}_{\Rightarrow}(T)$, as $a \Rightarrow x$ for some $a \in T$, it is $x \in T$.

It should be pointed out that, if T is finite, $\mathbb{C}_{\Rightarrow}(T)$ should not be also finite. Just consider $E = \mathbb{N}$, $\Rightarrow = \mathbb{N} \times \mathbb{N}$ and $T = \{1\}$: it is $\mathbb{C}_{\Rightarrow}(T) = \mathbb{N}$. Nevertheless, being E finite or $\Rightarrow$ finite, if T is finite so it is $\mathbb{C}_{\Rightarrow}(T)$.

Theorem 3.2. A relation $\Rightarrow$ is reflexive if and only if $T \subset \mathbb{C}_{\Rightarrow}(T)$ for any $\emptyset \neq T \subset E$.

Proof. If $\Rightarrow$ is reflexive, as $a \Rightarrow a$ for each $a \in T$, it is $a \in \mathbb{C}_{\Rightarrow}(T)$ and $T \subset \mathbb{C}_{\Rightarrow}(T)$. Reciprocally, for any $a \in E$ it is $\{a\} \subset \mathbb{C}_{\Rightarrow}(\{a\})$, and $a \Rightarrow a$.

Corollary. A reflexive relation $\Rightarrow$ is a T-conditional iff $T = \mathbb{C}_{\Rightarrow}(T)$.

Theorem 3.3. A relation $\Rightarrow$ is transitive if and only if $\mathbb{C}_{\Rightarrow}(\mathbb{C}_{\Rightarrow}(T)) \subset \mathbb{C}_{\Rightarrow}(T)$, for any non-empty subset T of E.

Proof. If $a \Rightarrow b$ and $b \Rightarrow c$, from $b \in \mathbb{C}_{\Rightarrow}(\{a\})$ and $c \in \mathbb{C}_{\Rightarrow}(\{b\})$ it follows $c \in \mathbb{C}_{\Rightarrow}(\{b\}) \subset \mathbb{C}_{\Rightarrow}(\mathbb{C}_{\Rightarrow}(\{a\})$, and $a \Rightarrow c$. Reciprocally, being $\Rightarrow$ transitive, if $x \in \mathbb{C}_{\Rightarrow}(\mathbb{C}_{\Rightarrow}(T))$ it exists some $b \in \mathbb{C}_{\Rightarrow}(T)$ such that $b \Rightarrow x$; but it also exists some $c \in T$ such that $c \Rightarrow b$: then $c \Rightarrow x$, or $x \in \mathbb{C}_{\Rightarrow}(T)$.

Corollary. If $\Rightarrow$ is transitive, it is a $\mathbb{C}_{\Rightarrow}$(T)-conditional for any $\emptyset \neq T \subset E$.

Corollary. If $\Rightarrow$ is a preorder, it is a $\mathbb{C}_{\Rightarrow}$(T)-conditional and $T \subset \mathbb{C}_{\Rightarrow}(T)$ for any $\emptyset \neq T \subset E$.

Corollary. A reflexive relation $\Rightarrow$ is transitive iff $\mathbb{C}_{\Rightarrow}(\mathbb{C}_{\Rightarrow}(T)) = \mathbb{C}_{\Rightarrow}(T)$ for each $T \subset E$, $T \neq \emptyset$.

Theorem 3.4. Mapping $\mathbb{C}_{\Rightarrow}$ is a Tarski's Consequences Operator [3] iff $\Rightarrow$ is a preorder.

Proof. Is an inmediate consequence of theorem 3.2 and 3.3. Then, being $\Rightarrow$ a preorder, it has complete sense to say that b is a consequence of a, each time that $a \Rightarrow b$.

Theorem 3.5. If $\Rightarrow$ is a preorder, for any $\emptyset \neq T \subset E$, it is $\mathbb{C}_{\Rightarrow}(T)$ the smallest subset of E that contains T and for which $\Rightarrow$ is a conditional.

Proof. The set $\mathcal{C} = \{X \subset E; T \subset X$ and $\Rightarrow$ is and X-conditional$\}$ is not-empty because $E \in \mathcal{C}$. Let it be

$$\overline{T} = \bigcap_{C \in \mathcal{C}} X.$$

It is $T \subset \overline{T}$; then $\mathbb{C}_{\Rightarrow}(T) \subset \mathbb{C}_{\Rightarrow}(\overline{T})$. It is $\overline{T} \subset \mathbb{C}_{\Rightarrow}(\overline{T})$; if $x \in \mathbb{C}_{\Rightarrow}(\overline{T})$ it exists some $a \in \overline{T}$ such that $a \Rightarrow x$ and, as $\Rightarrow$ is and $\overline{T}$-conditional, $x \in \overline{T}$ and $\mathbb{C}_{\Rightarrow}(\overline{T}) \subset \overline{T}$; but as $\Rightarrow$ is a $\mathbb{C}_{\Rightarrow}(T)$-conditional it is $\overline{T} \subset \mathbb{C}_{\Rightarrow}(T)$ and, finally, $\overline{T} = \mathbb{C}_{\Rightarrow}(T)$.

Corollary. Given a preorder $\Rightarrow$ on E and a subset $T \subset E$, $T \neq \emptyset$, it suffices to extend T to $\mathbb{C}_{\Rightarrow}(T)$ for having that $\Rightarrow$ is a $\mathbb{C}_{\Rightarrow}(T)$-conditional, provided that $\Rightarrow$ does not to be a T-conditional.

Then, each time that $a \Rightarrow b$ for both a and b in T, we can say that b is a consequence of a. It should be remarked that, if $\Rightarrow$ is not a preorder it can be extended to the preorder $\Rightarrow_{rt}$ for which follows the last assertion. In any case, if $\Rightarrow$ is not a preorder, but it is a

T-conditional, as $\Rightarrow \subset \rightarrow_T$, it follows

$$C_{\Rightarrow}(T) \subset C_{\rightarrow A}(T),$$

and the each $x \in C_{\Rightarrow}(T)$ can be considered as a consequence of T.

Theorem 3.6. Given $(E, \Rightarrow)$ and a function $\mu : E \rightarrow [0,1]$ such that "If $a \Rightarrow b$, then $\mu(a) \leq \mu(b)$", then, for each $\epsilon \in (0,1]$, is $\Rightarrow$ aμ^{-1} $([\epsilon, 1])$- conditional.

Proof. If $a \in \mu^{-1}([\epsilon,1])$ and $a \Rightarrow b$, it is $\epsilon \leq \mu(a) \leq 1$ and $\mu(a) \leq \mu(b) \leq 1$, then $\epsilon \leq \mu(b) \leq 1$ and $b \in \mu^{-1}([\epsilon,1])$.

For example, if E is a Boolean Algebra and p is a probability on E, as $a \leq b$ implies $p(a) \leq p(b)$, the partial order $\leq$ is a P_ϵ-conditional, being

$$P_\epsilon = \{x \in E : \epsilon \leq p(x) \leq 1\},$$

for each ϵ in (0,1].

The last theorem opens the door to exactify some parts of Approximate Reasoning [5].

References.

[1] Hans Reichenbach, "Elements of Symbolic Logic", Dover, New York, (1980).

[2] John P. Cleave, "The Notion of Logical Consequence in the Logic of Inexact Predicates". *Zeitsch. f. math. Logic und Grundlagen d. Math.* 20, 307-324 (1974).

[3] J.L. Castro and E. Trillas, "Sobre preórdenes y operadores de consecuencia de Tarski", *Theoria* (1989), 11, 419-425.

[4] S. Körner, "Experience and Theory", Routledge and Kegan Paul, London (1969).

[5] E. Trillas, "Some reflections on Inexact Inference" (preprint, 1991).

Instituto Nacional de Técnica Aeroespacial
Departamento de Inteligencia Artificial
Universidad Politécnica de Madrid
Madrid, Spain.