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ON THE NOTION OF FUZZY SET[1]

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ABSTRACT

Many discussions have been made on the problem of

i) What are Fuzzy Sets?

since the origin of the theory. Due to the structure of Fuzzy Sets the first impression that many people have is that Fuzzy Sets are the distribution of a probability. Recent developements of many theory of uncertainty measures (belief functions, possibility and fuzzy measures, capacities) can make also think that a Fuzzy Set is the distribution of an uncertainty measure. Other problems arising inside the theory of Fuzzy Sets itself compel to seek to a clear answer to the problem *i*).

In order to give an answer to the problem let us begin discussing some aspects of the theory of Fuzzy Sets. The first thing to be noted is that several different definitions of Fuzzy Sets exist:

Definition. a) (Zadeh [16]) Fixed a set X, a Fuzzy Set on X is a function with domain X and range contained in the interval [0,1], that is a function:

$$f : X \to [0, 1].$$

b) (Brown [2]) Fixed a set X, a Fuzzy Set on X is a function with domain X and range contained in a Boole algebra B, that is a function:

$$f : X \to B.$$

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c) (Goguen [8]) Fixed a set X, a Fuzzy Sets on X is a function with domain X and range contained in a partially ordered set V, that is a function:

$$f : X \rightarrow V.$$

d) In every case the range of the function is called the valuation set of the Fuzzy sets.

For simplicity's sake we will adopt the following

Convention. From now on we will call an object satisfying the preceeding definition (point a) or b) or c)) a naive Fuzzy Set (written briefly F-Sets) and we will call naive Fuzzy Sets theory the theory developed of F-Sets.

Moreover with "fuzzy" we indicate the adjective for the kind of vagueness known as "fuzziness".

Goguen's is the most general of the definitions given above; in every case, note that these definitions have been given within classical set theory. However, F-Sets want to be more general than classical sets that it is not suitable to describe vagueness, but till they are confined into classical set theory they cannot describe reality better than classical sets as they claim to do. If we want to overcome problems in the description of reality arising from classical set theory we must proceed in a different way from that which has been done so far: we must build a theory of fuzzy objects "ab initio", that is, we must not, in some way, build fuzzy objects within classical sets (or some other pre-existing theory) because the objects that are built inside a theory inherit every limits of the theory itself.

The only way in which a theory of fuzzy objects can be built is to develop an axiomatic theory of fuzzy classes: this can be done for example in a similar way to what has been done for the classical set theory by Zermelo and Fraenkel (see [11]).

The first attempt in this direction was published in 1974 by Chapin ([3], see also the continuation [4]) which was followed later on by [15], [9], [11], [12].

We do not want to take into consideration here axiomatic theories and we will examine problem i) speaking as little as possible of its axiomatic developement; nevertheless some remarks on axiomatic theories will be necessary to focus the argument as well as possible. For discussions about the axiomatic theories see [12] and [13], see also the special issue of the review "Fuzzy Sets and Systems, 1988, n. 1" and the paper quoted there for discussions on F-Sets.

Apart from [11] and [12] all the mentioned theories have as a goal the axiomatization of objects that are as near as possible to naive F-Sets and do not take into consideration the following problem: *ii*) do naive F-Sets perfectly translate real objects?

There are people asking more radical questions: for example [7] asks:

i) is it useful to model the fuzziness of our perception with F-Sets?

and more in general

ii) is it useful to model fuzziness?

We answer quite acritically "yes" to ii) and we refer the reader to the mass of papers on applications of F-Sets. We will try now to answer question *ii*), or i) of [7] that if is almost the same. Then the answer to *i*) will follow.

If we want to kave an idea of the vagueness known as "fuzziness" the best way is to quote [16]:

"More often than not, the classes of objects encountered in the real world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1."

That is, it seems that real fuzzy objects are structured as naive F-Sets (in part, at least): that is they satisfy the following property

1) every object has membership degrees and a valuation object.

We are going to show some other properties of real fuzzy objects: we must say right now that naive F-Sets do not satisfy the properties that we will show in the following. We will then deduce that F-Sets do not describe perfectly real fuzzy objects as they are supposed to, even if they are very useful objects.

Another property that seems to be satisfied by real objects is:

2) every object has its specific valuation object.

That is: different objects can have different valuation objects.

Indeed, why must every membership degree of every object belong to the same valuation set? Or in other words, why must membership to two different fuzzy objects be measured in the same way?

Criticizing [1], French notes that ([7] p. 34): (remember that it is a common practice to indicate the membership function of an F-Set A with μ_A)

"Namely, they ([1]) assume that μ_A and μ_B are measured on the same scale. If A is the Fuzzy Set short, and B the Fuzzy Set fat, then they assume that it is meaningful to write for a particular man x:

$$\mu_A(x) = \mu_B(x),$$

i.e., "x is as short as he is fat" What does this expression mean operationally? I have no conception of 'equal shortness and fatness'. The assumption that all membership functions are commensurate is not made by these authors alone; it is common to all Fuzzy Set theories. It has received negligible discussion. It means, among other things, that there is a reference scale of 'membership of a set' in the same way that the auxiliary experiment forms a reference scale in the measurement of subjective probability."

It seems that no one (apart from [7] and [5] for categorical considerations) has taken into consideration F-Sets with the valuation set that can be different from time to time; this because no one has thought that there is something more suitable for descibing reality than F-Sets with a fixed valuation set (as is commonly accepted). F-Sets with different valuation sets can be introduced not only to have a more general definition of F-Sets, but because real objects seem to have different valuation sets; as noted. With the following examples we intend to show that real objects actually satisfy the second property:

a) Take the fuzzy class T ='tall man': since height is measured in meters (i.e. it is a linear quantity) we are compelled to think that T has a linearly ordered valuation set (for example it can be [0,1]); in this case T seems to be suitably represented by an F-Set (that is a classical function) of the kind:

$$f : X \to [0, 1].$$

Stronger than the reason above there is another one which compels us to assume a linearly ordered valuation set in this case: this is the fact that we can always compare two men, x and y, and establish when 'x is taller that y' (or not), moreover we can do this exactly; since the membership degree of a man x in T, $\mu_T(x)$, represents how such x satisfies the condition of 'being tall' (or in other words $\mu_T(x)$ gives an indication of how tall x is) it seems (in this case) that we can always compare membership degrees: that is, again, we have a linearly ordered valuation set.

b) Take now the fuzzy class I='cumbersome package': contrary to the preceeding case, we cannot always say that one package is more (or less) cumbersome than another: for example we could take a very tall and very narrow package and, on the contrary, a short and very large package; these two packages are not comparable from the point of view of encumbrance since their encumbrance depends upon two different dimensions: the largeness on the one hand and the height on the other. In other words encumbrance is a multidimensional property and so the valuation object of the fuzzy class should be a multidimensional set to take into consideration the multidimensionality of the class: for example if the fuzzy class T has a linearly ordered valuation set as [0,1], then every package would be comparable with every other package from the point of view of encumbrance, while we have shown that this is not the case.

In Zadeh's F-Sets (and also in Brown's and Goguen's) we note that the valuation set is a crisp (well-defined, classical) set, while the F-Set is (represents) a fuzzy object. Moreover, in the naive formulation of F-Sets a crisp comparison relation between membership degrees is given. But it seems more suitable to have a valuation set which is itself fuzzy and have a fuzzy comparison relation on degrees, as can be seen from the following examples.

In natural language we have such expressions as:

"John is quite tall"

"Mary is much taller than Jane"

"Paul is nearly as tall as Peter".

All these examples signify that either the membership degree ("quite tall") or the comparison between membership degrees ("much more than", "nearly as") are themselves fuzzy objects: at least they are as fuzzy as the concept height itself.

That is, real fuzzy objects seem to satisfy the following two conditions too:

3) the valuation set is fuzzy;

(in particular it has membership degrees and a valuation set), and

4) the comparison relation between membership degrees is fuzzy.

In the several formulations of F-Sets (Zadeh's, Brown's or Goguen's), since in every case they are defined inside classical set theory, the valuation set is a crisp set, made up of crisp objects, with a crisp comparison relation between its members.

To try to overcome this situation F-Sets of type n have been introduced (see [6]): in these objects membership degrees are F-Sets, but the valuation set is again a crisp set with a crisp comparison relation.

Note that in the naive theory of F-Sets the comparison relation is used not only to compare membership degrees but also to define set-operations and other set-notions; for example we remember that inclusion in F-Sets is defined (in the case of Zadeh's F-Sets) as follows

Definition. Given two F-Sets $A : X \to [0,1]$, and $B : X \to [0,1]$,

$$A^{c}_{\text{Zadeh}}B \quad \text{if and only if} \quad (\forall x \in X)\mu_A(x) \leq \mu_B(x).$$

Note that the preceeding notion of inclusion requires that

a) the valuation set is fixed and unique for all sets;

b) the comparison relation is crisp.

If (as we suppose) real objects satisfy the properties we have established, the preceeding notion of inclusion (or a similar one) cannot be given on real objects, and cannot be given on the mathematical transposition of real objects in an axiomatization satisfying the preceeding properties beeing suitable only on Zadeh's F-sets.

A suitable notion of inclusion in every case can be established in a axiomatization. In fact, inclusion can be obtained by the connective $\Longrightarrow$ (implication, if .. then ..), as can be deduced by the following example:

take the two classes

L="long package" and I="cumbersome package";

It seems that the first class is contained in the other one: indeed it is true that:

if a package is long then it is cumbersome.

Note that if we deduce in some way two F-Sets of Zadeh, $\mathcal{L}$ and $\mathcal{F}$, by these two properties it may occur that $\mathcal{L}$ is not contained in $\mathcal{F}$ following the definition of Zadeh's inclusion. For example a package that is 2 meters long will have membership 1 in $\mathcal{L}$, but if it is very small in the other dimensions then it will have membership in $\mathcal{F}$ (strictly) smaller than 1.

Another problem in F-Sets is the following:

take for example the fuzzy class "tall man" and suppose that it is represented by the F-Set $f : X \to [0,1]$; then we note that there are different ways of treating objects depending on whether the objects belong to X or not: there are the following three cases:

a) $x \notin X$

b) $x \in X$ & $f(x) = 0$ (*)

c) $x \in X$ & $f(x) > 0$.

Remember that, on the contrary, in the classical set theory there are the following two cases only:

a) $x \notin X$ (**)

b) $x \in X$

Commonly case (*),b) is interpreted as: x does not satisfy the condition "being tall". We also consider this to be the correct interpretation of the case; but then, how can the case (*),a) be interpreted? The only way in which case (*),b) can be interpreted seems to be the following:

"x is not of interest for the property 'being tall man'".

Then the set X must be interpreted as the "universe of interesting things". This interpretation of the three cases is confirmed by applications: for example in [10] where the control of a traffic junction is studied with F- Sets, the F-Sets "long queue" and "short queue" are assumed to be F-Sets with domain the (crisp) set

$$X = \{4, 5, 6, \ldots, 31, 32\}.$$

The domain of the two F-Sets is not $\mathbb{N}$ (the set of natural numbers) but is assumed to be a very small subset of it. This subset of $\mathbb{N}$ is such that a number n (of car waiting at the crossroad) is in X if and only if n is an interesnting number (of cars waiting at the crossroad): indeed 0 (cars), 1 (car), 2 (cars), 3 (cars) are not in X since they represent such a small queue that it is not interesting for the direction of the crossroad (such a small number of cars at the crossroad does not cause trouble). In the same way every number greater than 32 is not in X since it represents a queue that can be considered (for the direction of the traffic junction) as long as a 32 car queue (a number of cars greater than 32 causes the same trouble as 32 cars at the crossroad).

There is another problem about F-Sets (of Zadeh): there exist several different interpretations of the linguistic connectives and, or, not; note that this does not happen in classical set theory where the connectives are respectively interpreted:

and in the intersection,

or in the union,

not in the complement.

On the contrary in F-Sets there are several definitions of the set-operations; for example, having two F-Sets A and B, the connective and can be translated using the operators

a) infimum obtaining the intersection of A and B, $A \cap B$;

b) product obtaining the product of A and B, $A \cdot B$;

c) max $(1, x + y - 1)$ obtaining the bold intersection of A and B, $A \cap B$.

If one takes the two properties "tall man" and "intelligent man", using the connective and we obtain the (only one) property:

"tall and intelligent man",

If T and I are two F-Sets representing the two properties, we do not see why the unique property "tall and intelligent man" could be translated in the following three different ways

$$T \cap I, \qquad \text{or} \qquad T \cdot I, \qquad \text{or } T \cap I.$$

The same thing happens for the disjunction (and union).

These observations compel us to assume that real objects satisfy the following condition:

5) there exists only one definition for every set-operation.

Naive F-Sets (in every definition) do not satisfy conditions 2)–4) as can be easily seen:

for condition 2): the valuation set is fixed for every F-Set;

for condition 3): the valuation set is a crisp set;

for condition 4): the comparison relation between membership degrees is crisp;

Zadeh's F-Sets do not even satisfy property 5). Apparently property 5) is satisfied by the axiomatizations of [15] and [3] since only one definition for every set-operation is assumed, but other definitions can be established.

Let us now make some further observations about the axiomatizations of [3] and [15] ([9]'s paper is too short to be discussed): objects axiomatized in these theories do not satisfy these conditions: in fact the valuation set is fixed and unique for every object (even if it is not explicitly written).

These axiomatizations do not satisfy properties 3) and 4) either: they have a crisp comparison relation on membership degrees and this make them fail to satisfy 3). This also means that (in some way) they do not satisfy 4) either since they have then a crisp structure on the valuation set; therefore the valuation set is not so fuzzy as the other objects.

These axiomatizations do not satisfy the three cases either, showing in this way that the sets axiomatized are not so similar to naive F-Sets as they are supposed to.

Let us see what happens in everyday language to understand which is the behaviour of real objects with respect to the problem of the three cases of (*): if we take a property, say "tall man", and we group together in a collection the men satisfying this property obtaining the collection *tall man*, in our opinion, the men that are not tall are not considered as elements of the colleciton (with any membership degree). Our opinion is that when we obtain by abstraction a collection of objects satisfying a fixed property, only the objects that satisfy the property at least a little are grouped together in the collection; that is, an element of a real fuzzy object can have only a "positive" membership degree in the collection (in this direction see the position of [14]).

Now we want to answer the principal question: *i)* "what are F-Sets?".

In order to do this we should quote ([16] p. 339):

"A fuzzy class A in X is *characterized* by a *membership* function $f_A(x)$, which associates with each point in X a real number in the interval $[0,1]$, with the value of $f_A(x)$ representing the grade of membership of x in A".

Even if Zadeh's aim is to emphasize that F-Sets are the generalization of the concept of characteristic function, it seems that real objects are one thing and mathematical objects another thing that *characterize* (in Zadeh's words, but we prefer to say *represent*) real objects; this dichotomy appears in some other papers of Zadeh and in Brown, but later disappears (see Goguen's paper or [6]) and now F-Sets are defined as we have defined them at in the beginning of this paper.

We too think that there is this dichotomy between real objects and F-Sets, and we say more explicitely that F-Sets (Zadeh's or Brown's or Goguen's) are "*representations*" (better that "*characterizations*") of real objects.

In general sets (classical or fuzzy) are the extensional representation of a property: that is they are the representation of a collection. Now, depending on the kind of representation of this collection (and of the reality) we have in mind, we use a classical representation or a non-classical one: that is a classical set or a non-classical set (fuzzy in general). In the case of F- sets, due to the fact that this representation uses classical sets we can also speak of "*approximation* of real objects" because an approximation of an object uses well known objects to be carried on, classical sets in this case.

Real fuzzy objects have a fuzzy structure and (in particular) they satisfy all the properties 1)–5), but, in many cases, the valuation set of the objects is too difficult to be used in applications (even to be grasped in most cases). Then it is always forced to be the interval [0,1] (or some other crisp set). Our incapacity to see the valuation set and its structure also forces the comparison relation between membership degrees to be the natural order, on [0,1].

In this direction *I* must say that the choice of [0,1] as valuation set, even if natural from the mathematical point of view and the most useful, caused and is still causing misunderstandings because if we define F-Sets as funcitons of the form $F : X \to [0, 1]$ it is very natural to think that F-Sets are only a distribution of a probability, even if there is a big conceptual difference between uncertainty measures and sets, between the description of a human expectation and an extensional description of reality. This misunderstanding would not have been appeared if F-Sets would have been defined from the beginning as in Goguen's definition.

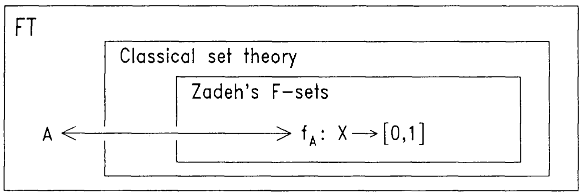
It is natural then that there exist several systems of fuzzy representations: that is there exist several structures on F-Sets (several systems of set operations). In practice: given a problem, we choose to represent real collections with F-Sets (generally Zadeh's) making a restriction on the objects involved in the problem: the collection of the most interesting objects will constitute the domain of the F-Sets. At the same time we choose a "law of coherence" that must be fulfilled by the representations themselves: this law forces a collection of set-operations (intersection, union, complement,...) on the representations.

It must be studied which can be the most convenient system of set-operations on F-Set and, in our opinion, a lot of the papers appeared till now are just studies in this direction (see some paper of the issue of the revue Fuzzy Sets and Systems 1988, n. 1, and some of the papers quoted there).

Even if F-Sets do not perfectly translate real objects, their utility is proved by the large number of succesful applications: we also think that (at the moment) they are the best way of translating real fuzzy collections and that better translations are interesting and possible only at a theoretical stage.

By the way an axiomatic treatement of F-sets can be interesting for other aspects: having an axiomatic theory FT of F-sets in which the axiomatized objects satisfy the same properties of the reality we can say that we have real fuzzy objects inside the theory. Inside FT, classical set theory and the F-sets of Zadeh should be built and then we can study the relations between a real fuzzy object A and its representation made by an F-set

$f_A : X \to [0,1]$.



In this way we can study also the systems of operations of F-sets and we will try to choose which is the best system of operations on F-sets. By the way this is a future work.

References.

[1] Belmann, R.E., Giertz (1973) "On the Analytic Formalism of the Theory of Fuzzy Sets", *Inf. Sci.* 5, 149-157.

[2] Brown, J.G. (1971) "A Note on Fuzzy Sets", *Inf. Control* 18, 32-39.

[3] Chapin, E.W. (1974) "Set Valued Set Theory: Part I", *Notre Dame J. Formal Logic*, 4, 619-634.

[4] Chapin, E.W. (1975) "Set Valued Set Theory: Part II", *Notre Dame J. Formal Logic*, 5, 255-267.

[5] Di Nola, A., Gerla, G. (1984) "Lattice Valued Algebras", *Pubblicazioni dell'Istituto di Matematica "R. Caccioppoli"*, Napoli.

[6] Dubois D., Prade, P. (1980) *Fuzzy Sets and Systems. Theory and Applications*, Academic Press, New York.

[7] French, S. (1984) "Fuzzy Decision Analysis: Some criticisms", *TIMS/Studies in Management Sciences*, 20, 29-44.

[8] Goguen, J.A. (1967) "L-Fuzzy Sets", *J. Math. Anal. Appl.* 18, 145-174.

[9] Novàk, V. (1980) "An attempt at Godel-Bernays-like Axiomation of Fuzzy Sets", *Fuzzy Sets and Systems*, 3, 323-325.

[10] Pappis, C.P., Mamdami, E.M. (1977) "A Fuzzy Logic Controller for a Traffic Junc-

tion", *IEEE Trans. on Sys. Man and Cyb.* 7, n. 10, 707-717.

[11] Prati, N. (1988) "An Axiomatization of Fuzzy Classes", *Stochastica*, XII-I, 65-78.

[12] Prati, N. (1988) *Una Teoria degli Insiemi Fuzzy ed Alternativi.* Tesi di Dottorato di Ricerca in Matematica, Università di Pisa.

[13] Prati, N., "On the comparison between Fuzzy Sets axiomatizations", in printing.

[14] Toth, H. (1987) "From Fuzzy-Set Theory to Fuzzy Set-Theory: Some critical Remarks on Existing Concepts" *Fuzzy Sets and Systems*, 23, 219- 237.

[15] Weidner, A.J. (1981) "Fuzzy Sets and Boolean-Valued Universes". *Fuzzy Sets and Systems* 6, 61-72.

[16] Zadeh, L.A. (1965) "Fuzzy Sets", *Inf. Contr.* 8, 338-353.

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