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ON THE SPECTRAL REPRESENTATION OF THE SAMPLING CARDINAL SERIES EXPANSION OF WEAKLY STATIONARY STOCHASTIC PROCESSES

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ABSTRACT

The spectral representation of the sampling cardinal series expansion (SCSE) of non-band-limited weakly stationary scalar and vector stochastic processes and their correlation functions are derived. The upper bound of the mean-square aliasing error is given for vector processes.

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I. Introduction.

A weakly stationary (WS) mean-square continuous process $\{X(t) \mid t \in \mathbb{R}\}$ with variance $DX(t) = E|X(t)|^2 = \sigma^2$ is said to be band- limited to frequency ω if its spectral measure Φ satisfies the condition $\Phi(]-\infty, -\omega]) = \Phi([\omega, \infty[) = 0$. The spectral distribution function is $F(\lambda) = \Phi(]-\infty, \lambda])$; so $X(t)$ is band-limited there if $F(\lambda) = 0$ for all $\lambda \leq -\omega$ and $F(\lambda) = \sigma^2$ for all $\lambda > \omega$.

The autocovariance function $X(t) = EX(t)X^*(0)$ of a zero mean process $X(t)$ has an integral representation in the form

$$K(t) = \int_{-\omega}^{\omega} e^{it\lambda}\, dF(\lambda). \tag{1.1}$$

We note that the endpoints $\pm\omega$ are required to be continuity points of F.

Such a process possesses an integral representation in the form $X(t) = \int_{-\omega}^{\omega} e^{it\lambda}\, dZ(\lambda)$. Here is $Z(\lambda)$ (the spectral process of $X(t)$) a process with orthogonal increments.

The stochastic sampling theorem is under foregoing conditions

$$X(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{k=-n}^{n} X(k\pi/\tilde{\omega})\text{sinc}(\tilde{\omega}t - k\pi) \tag{1.2}$$

for arbitrary $\tilde{\omega} \geq \omega > 0$, $\text{sinc}(x) \triangleq x^{-1}\sin(x)$. [1], [2].

For the non-band-limited WS stochastic processes holds a similar result. Namely if $\omega_m/n \to 0$ as $n, m \to \infty$, then (1.3) holds:

$$X(t) = \underset{n,m\to\infty}{\text{l.i.m.}} \sum_{k=-n}^{n} X_m(k\pi/\omega_m)\text{sinc}(\omega_m t - k\pi) \tag{1.3}$$

where $X_m(t) = \int_{\mathbf{R}} e^{it\lambda} 1_{]-\omega_m,\omega_m[}(\lambda)\ dz(\lambda)$, $\{\omega_m\}_1^\infty$ is a positive increasing real sequence divergent to ∞, $1_A(\lambda)$ denotes the characteristic function of the set A, [8].

In the non-band-limited case, we denote the sampling cardinal series expansion for a given choice of bandwidth $\omega > 0$ by $X_a(t)$. So

$$X_a(t) = \sum_{-\infty}^{\infty} X(n\pi/\omega)\text{sinc}(\omega t - n\pi).$$

When only the randomness of the process is considered, then $X(t) - X_a(t)$ is not WS in general, because the so-called aliasing errorr (in the mean- square sense)

$$a_X(t) = E|X(t) - X_a(t)|^2$$

depends on t; practically $a_X(n\pi/\omega) = 0$, since $X_a(t)$ interpolates $X(t)$ at the sampling points, [4].

Brown has shown, [4], based on Weiss's theorem, [9], that

$$a_X(t) \leq 4(\sigma^2 - F(\omega) + F(-\omega)). \tag{1.4}$$

In the current paper we prove the explicit formulae of the spectral representation of $X_a(t)$, its autocovariance function $K_a(t) = EX_a(t+s)X_a^*(s)$, and generalize this results to the multidimensional WS non- band-limited processes. Based on the spectral representation of the sampling cardinal series expansion (SCSE) we give a rigorous stochastic terminological proof of the Brown's upper bound (1.4) and derive related vector aliasing error analysis results. The Lloyd's result on the spectral statement representation of the determination of a WS process $\xi(t)$ by its samples can be formualted as follows, [7].

Let $\wedge$ be the support of the spectral distribution function $G(\lambda)$ of a process $\{\xi(t) \mid t \in \mathbb{R}\}$. If $\wedge =]-\omega, \omega[$ and all translates of $\wedge$ (i.e. $\{\wedge - 2k\omega \mid k \in \mathbb{Z}\}$) are mutually disjoint then

$$\xi(t) = \int_{\mathbb{R}} (e^{it\lambda})_{2\omega} d\xi(\lambda). \tag{1.5}$$

Then easily follows that

$$K_{\xi}(t) = \int_{\wedge} (e^{it\lambda})_{2\omega} dG(\lambda), \tag{1.6}$$

where $(e^{it\lambda})_{2\omega}$ is defined at the beginning of section II (see [2]).

Thus Lloyd eliminated the continuity problem at the "critical" endpoints $(2k+1)\omega$ (such that the spectral distribution function may possesses jumps at $(2k+1)\omega$) with the open support technique, or supposing that the sampling theorem there holds.

Let $\wedge \triangleq]-\omega, \omega]$. We shall prove that (1.5) and (1.6) hold iff the spectral distribution function of the observed non-band-limited full-spectrum process is continuous at the points $(2k+1)\omega$, $k \in \mathbb{Z}$. In the section III we introduce the spectral representation of the cross-covariance function of stationarily correlated SCSE's. This results are generalized to the multidimensional weakly stationary processes and its SCSE's in the section IV. Obviously the spectral representation of a band-limited WS stochastic process is also given with the aid of our results because $(e^{it\lambda})_{2\omega} \equiv e^{it\lambda}$ on $]-\omega, \omega]$.

II. Spectral representation of SCSE's.

Let $\{X(t) \mid t \in \mathbb{R}\}$ be a non-band-limited stochastic process on the probability space

$(\Omega, \mathcal{F}, P)$. For each real fixed t define $(e^{it\lambda})_{2\omega}$ as the periodic extension (with period 2ω) of the function $e^{it\lambda}$ from the interval $]-\omega, \omega]$ to the entire real axis.

The complex Fourier-series of $(e^{it\lambda})_{2\omega}$ is

$$(e^{it\lambda})_{2\omega} = \sum_{-\infty}^{\infty} \exp(in\lambda\pi/\omega)\text{sinc}(\omega t - n\pi). \tag{2.1}$$

The convergence in (2.1) is everywhere with respect to λ, for all $t \in \mathbf{R}$. Because $(e^{it\lambda})_{2\omega}$ is continuous on $]-\omega, \omega]$ it satisfies the well-known Dirichlet-condition. Hence, the convergence in (2.1) is also pointwise except at the points $\lambda = (2k+1)\omega$, k integer. From the Fourier-series theory, the sequence of partial sums on the right-hand-side of (2.1) converges to $\cos(\omega t)$ at the points $\lambda = (2k+1)\omega$.

The spectral representations of the process $X(t)$ and of its correlation function are

$$X(t) = \int_{\mathbf{R}} e^{it\lambda}\, dZ(\lambda); \qquad K(t) = \int_{\mathbf{R}} e^{it\lambda}\, dF(\lambda)$$

respectively. The masses of the spectral distribution function $F(\lambda)$ at the points $(2k+1)\omega$ are denoted by F_k, i.e. $F_k = F((2k+1)\omega+) - F((2k+1)\omega)$.

Lemma 1. Let r be an arbitrary positive integer an $N \geq 2$. Then,

a) for $\lambda \in]-\omega, \omega]$

$$\sum_{-N}^{N} \exp(in\pi\lambda(\omega)\text{sinc}\ (\omega t - n\pi) + \theta(N^{-r}\ln N) = e^{it\lambda}$$

b) for $\lambda \in \{\omega, \omega\}$

$$\sum_{-N}^{N} \exp(in\pi\lambda(\omega)\text{sinc}\ (\omega t - n\pi) + \theta(N^{-1}) = \cos(\omega t).$$

Setting $\alpha = 1$ if $\lambda \in]-\omega, \omega[$ and $\alpha = 0$ if $\lambda \in \{-\omega, \omega\}$this lemma can be stated as

$$\sum_{-N}^{N} \exp(in\pi\lambda/\omega)\text{sinc}\ (\omega t - n\pi) + \theta(N^{-\alpha(r-1)-1}\ln^{\alpha} N) = \alpha e^{it\lambda} + (1-\alpha)\cos(\omega t) \tag{2.2}$$

Proof. Consider a 2ω-periodic, r-fold derivable function $f(\lambda)$ such that $|f^{(r)}(\lambda)| \leq M_r$. Then the residual $R_N(f)$, of the symmetric complex Fourier expansion of $f(\lambda)$ on $(-\omega, \omega)$ is upper-bounded, namely:

$$|R_N(f)| \leq AM_r(\omega/\pi)^r N^{-r} \ln N. \tag{2.3}$$

Here, A is an absolute constant. This Bernstein's result is detaily treated in [5], where $A = 2 + (1 + \ln \pi)/\ln 2$ is suggested.

The residual of the Fourier expansion of $e^{it\lambda}$ on $[-\omega, \omega]$ is $R_N(e^{it\lambda}) = \sum_{|r|>N} \exp(in\pi\lambda/\omega)\text{sinc}(\omega t - n\pi)$. From (2.3) it follows:

$$|R_N(e^{it\lambda})| \leq A(\omega|t|/\pi)^r N^{-r} \ln N \tag{2.4}$$

for all $\lambda \in]-\omega, \omega[$.

Because $\sum_{-N}^{N} \exp(in\pi\lambda/\omega)\text{sinc}(\omega t - n\pi) = \sum_{-N}^{N} (\omega t - n\pi)^{-1} \sin(\omega t)$ at the points $\lambda = \pm\omega$ for sufficiently large N we obtain

$$\begin{aligned} |\cos(\omega t) - \sum_{-N}^{N} (\omega t - n\pi)^{-1} \sin(\omega t)| &\leq 2\omega|t| \sum_{-N}^{N} |\omega t - n\pi|^{-1} \\ &< 2\omega|t|\pi^{-2} \sum_{N+1}^{\infty} n^{-2} < 2\omega|t|\pi^{-2}/N \end{aligned}$$

and, consequently, the assertion follows from (2.3) and (2.4). ■

It is sufficient to use $r = 1$ in the evaluation (2.2). Then

$$\sum_{-N}^{N} \exp(in\lambda\pi/\omega)\text{sinc}(\omega t - n\pi) + O(N^{-1} \ln^{\alpha} N) = \alpha e^{it\lambda} + (1 - \alpha)\cos(\omega t). \tag{2.5}$$

Proposition 2.1. The SCSE $X_a(t)$ of a non-band-limited WS stochastic process $X(t)$ possesses a spectral representation

$$X_a(t) = \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} \, dZ(\lambda), \tag{2.6}$$

iff $F(\lambda)$ is continuous at the points $\lambda = (2k+1)\omega$, $k \in \mathbb{Z}$.

Proof. Let us take $X_N(t) = \sum_{-N}^{N} X(n\pi/\omega)\text{sinc}(\omega t - n\pi)$. Then

$$\begin{aligned}
B_n &= E\left|\int_{\mathbf{R}} (e^{it\lambda})_{2\omega}\, dZ(\lambda) - X_N(t)\right|^2 = \\
&= E\left|\int_{\mathbf{R}} \left((e^{it\lambda})_{2\omega} - \sum_{-N}^{N} \exp(in\lambda\pi/\omega)\text{sinc}(\omega t - n\pi)\right) dZ(\lambda)\right|^2 = \\
&= \int_{\mathbf{R}} \left|(e^{it\lambda})_{2\omega} - \sum_{-N}^{N} \exp(in\lambda\pi/\omega)\text{sinc}(wt - n\pi)\right|^2 dF(\lambda) = \\
&= \sum_k \int_{(2k-1)\omega+}^{(2k+1)\omega-} \left|e^{it(\lambda - 2kw)} - \sum_{-N}^{N} \exp(in\lambda\pi/\omega)\text{sinc}(\omega t - n\pi)\right|^2 dF(\lambda) + \\
&+ \sum_k \left|e^{it\omega} - \sum_{-N}^{N} \exp\left(in(2k+1)\pi\right)\text{sinc}(\omega t - n\pi)\right|^2 F_k. \qquad (2.7)
\end{aligned}$$

By the lemma 2.1 and (2.5) we have

$$B_N = \sin^2 \omega t \sum_K F_k + O(N^{-2}\ln^2 N).$$

On the other hand, B_N is uniformly upper-bounded above by $4\sigma^2$ and by the Lebesgue dominated convergence theorem we get

$$E\left|X_a(t) - \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} dZ(\lambda)\right|^2 = \lim_{N\to\infty} B_N = \sin^2 \omega t \sum_k F_k. \qquad (2.8)$$

The last evaluation completes the proof of the theorem. ■

In his paper [3] Brown has shown that for deterministic functions holds a similar result. But he supposed that if $A(\lambda)$ is a measurable and absolutely integrable function on the entire real axis and $A\widehat{}(t)$ the Fourier transform of $A(\lambda)$, then yields

$$A_a\widehat{}(t) = \sum_{-\infty}^{\infty} A_a\widehat{}(n\pi/\omega)\text{sinc}(\omega t - n\pi) = \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} A(\lambda) d\lambda, \qquad (2.9)$$

for each real t and $\omega > 0$, given (Lemma 2 in [3]).

It is also interesting to remark that Brown does not apply (2.9) in his investigations in aliasing error analysis of WS non-band-limited stochastic processes, [4].

Proposition 2.2. The autocovariance function $K_a(t)$ of the SCSE $X_a(t)$ is given by

$$K_a(t) = EX_a(t+s)X_a^*(s) = \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} dF(\lambda).$$

if F is continuous at the points $(2k+1)w$, $k \in \mathbf{Z}$.

Proof. As $Z(\lambda)$ is a process with orthogonal increments and $E\ dZ(\lambda)\ dZ^*(\mu) = \delta_{\lambda\mu}\Phi(d\lambda)$, from $\mathbf{R} = \cup_k](2k-1)\omega, (2k+1)\omega]$ it follows that

$$\begin{aligned} EX_a(t+s)X_a^*(s) &= E \int_{\mathbf{R}} (e^{i(t+s)\lambda})_{2\omega}\ dZ(\lambda) \int_{\mathbf{R}} (e^{-is2\mu})_{2\omega}\ dZ^*(\mu) \\ &= \sum_{k,j} \int_{I_k} \int_{I_j} e^{it(\lambda-2k\omega)} e^{is[(\lambda-\mu)-2(k-j)\omega]} E\ dZ(\lambda)\ dZ^*(\mu) \\ &= \sum_k \int_{I_k} e^{it(\lambda-2k\omega)} \Phi(d\lambda) \\ &= \int_{\mathbf{R}} (e^{it\lambda})_{2\omega}\ dF(\lambda), \end{aligned}$$

where is $I_k \triangleq](2k-1)\omega, (2k+1)\omega]$. The proof is complete. ∎

As $X_a(t) = \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} dZ(\lambda)$ it is easy to prove the relationship (1.4). Let $H(X)$ be the Hilbert-space of the process $X(t)$ (the L_2-closed linear subspace generated by the set $\{X(t) \mid t \in \mathbf{R}\}$) and $L_2(\mathbf{R}, dF) \triangleq \{\varphi \mid \int_{\mathbf{R}} |\varphi|^2 dF < \infty\}$. Thus, the explicit proof of the Brown's upper bound of the mean-square aliasing error exploating the well-known isometry between $H(X)$ and $L_2(\mathbf{R}, dF)$ is as follows

$$\begin{aligned} a_X(t) = E|X(t) - X_a(t)|^2 = E\left|\int_{\mathbf{R}} (e^{it\lambda} - (e^{it\lambda})_{2\omega}) dZ(\lambda)\right|^2 = \\ = \int_{\mathbf{R}} \left|e^{it\lambda} - (e^{it\lambda})_{2\omega}\right|^2\ dF(\lambda) \le 4(\sigma^2 - F(\omega) + F(-\omega)). \end{aligned}$$

As $X(t)$ is L_2-process ($DX(t) = K(0) = K_a(0) = DX_a(t) = \sigma^2$) it is not necessary to suppose that the spectral distribution function $F(\lambda)$ is absolutely continuous. Therefore we can weaken the condition in [4] on the existence of the spectral density of $X(t)$.

III. Stationarily correlated SCSE's.

Let $\{X(t) \mid t \in \mathbb{R}\}$, $\{Y(t) \mid t \in \mathbb{R}\}$ be WS non-band-limited stochastic processes on the same probability space $(\Omega, \mathcal{F}, P)$. In studying such processes we have to consider not only ordinary covariance functions, but also the so-called cross-covariance function of the zero mean processes $X(t)$ and $Y(t)$. So the cross-correlation function $K_{xy}(t) = EX(t)Y^*(0)$ has a spectral representation

$$K_{xy}(t) = \int_{\mathbb{R}} e^{it\lambda} dF_{xy}(\lambda) \tag{3.1}$$

where $F_{xy}(\lambda)$ is the so-called cross-spectral distribution function of the processes $X(t)$, $Y(t)$, i.e. $dF_{xy} = EdZ_x dZ_y^*$ and

$$X(t) = \int_{\mathbb{R}} e^{it\lambda} dZ_X(\lambda), \quad Y(t) = \int_{\mathbb{R}} e^{it\lambda} dZ_y(\lambda).$$

Let $X_a(t)$, $Y_b(t)$ be the SCSE's of processes $X(t)$ and $Y(t)$ associated with given bandwidths ω_x, ω_y respectively. Processes are stationarily correlated if there exists their cross-correlation function (3.1). Suppose in the sequel that at least one of the spectral distribution functions F_x, F_y is continuous at its sampling points $(2k+1)\omega_x$ and $(2k+1)\omega_y$ respectively. The main result of cross-covariance function of the SCSE's $X_a(t)$ and $Y_a(t)$ is given by

Proposition 3.1. Consider stationarily correlated processes $X(t), Y(t)$. Then $X_a(t)$ and $Y_b(t)$ are stationarily correlated iff $\omega_x = \omega_y$.

Proof. Suppose $X_a(t)$ and $Y_b(t)$ are stationarily correlated. Let us take $\omega_x = c\omega_y$. Since $(e^{it\lambda})_{2\omega_x} = (e^{itc\lambda})_{2\omega_y}$ we get

$$\begin{aligned} EX_a(t+s)Y_b^*(s) &= \int_{\mathbb{R}}\int_{\mathbb{R}} (e^{i(t+s)\lambda})_{2\omega_x}(e^{-is\lambda})_{2\omega_y} EdZ_x(\lambda)dZ_y^*(\lambda) \\ &= \int_{\mathbb{R}} (e^{i(ct+(c-1)s)\lambda})_{2\omega_y} dF_{xy}(\lambda). \end{aligned} \tag{3.2}$$

Now it is clear that (3.2) depends only on t if $c = 1$.

Because at least one of the functions F_x, F_y is continuous at its sampling points, from $|dF_{xy}(\lambda)|^2 \leq dF_x(\lambda)dF_y(\lambda)$ and (2.8) clearly follows (3.2).

The second part of the proof if obvious. ■

Remark. If ω_x are not equal to ω_y, we can choose a common bandwidth $\omega \geq \max\{\omega_x, \omega_y\}$, such that both considered processes have continuous spectral distribution functions at the poitns $(2k+1)\omega$, $k \in \mathbb{Z}$. In this case we do not lose any information on the nature of SCSE and with new common bandwidth ω the stationary correlation is possible.

Consequence 3.1. The cross-covariance function $K_{ab}(t) = EX_a(t+s)Y_b^*(s)$ of the SCSE's $X_a(t)$, $Y_b(t)$ possesses also a spectral representation. With the aid of the Proposition 3.1 we assume

$$K_{ab}(t) = \int_{\mathbb{R}} (e^{it\lambda})_{2\omega} dF_{xy}(\lambda).$$

Here is $\omega = \omega_x = \omega_y$.

Because $K_{ab}(t)$ is a correlation function therefore the following properties hold:

(i) $K_{ab}(t) = K_{ba}^*(-t)$

(ii) $|K_{ab}(t)|^2 \leq K_x(0)\ K_y(0) = \sigma_x^2 \sigma_y^2$.

Here is $DX(t) = \sigma_x^2$, $DY(t) = \sigma_y^2$.

IV. Multivariate SCSE's.

Analogously to the vector stochastic processes we consider now vector SCSE's. Observe a zero mean q-dimensional WS non-band-limited stochastic process $\{X(t) = (X_1(t), \ldots, X_q(t) \mid t \in \mathbb{R}\}$ with stationarily correlated coordinates. The coordinate-processes of $X(t)$ define on the common probability space $(\Omega, \mathcal{F}, P)$. Denotes $X_a^j(t)$ the SCSE of the j^{th} coordinate-process of $X(t)$ to given bandwidth w_j,

$j = 1, \ldots, q$. The role of the correlation function is playing by the correlation matrix $K_X(t) = (K_{jk}(t)]_{q\times q}$. Here is

$$K_{jk}(t) = EX_j(t)X_k^*(0) = \int_{\mathbf{R}} e^{it\lambda} dF_{jk}(\lambda),$$

where $dF_{jk}(\lambda) = EdZ_j(\lambda)dZ_k^*(\lambda)$. Now we define the multivariate SCSE $X_a(t) \triangleq (X_a^1(t), \ldots, X_a^q(t))$.

Proposition 4.1. The correlation matrix $K_a(t) = EX_a^T(t)XP_a^*(0) = (K_{aa}^{jk}(t))_{q\times q}$ of the SCSE $X_a(t)$ possesses the spectral representation

$$K_a(t) = \int_{\mathbf{R}} (e^{it\lambda})_{2\omega} dF(\lambda), \tag{4.1}$$

iff $\omega_1 = \ldots = \omega_q = \omega$ and all spectral distribution functions F_{jj} are continuous at the points $(2k+1)\omega$. Here, $K_{aa}^{jk}(t) \triangleq EX_a^j(t)(X_a^k(0))^* = \int_{\mathbf{R}}(e^{it\lambda})_{2\omega}dF_{jk}(\lambda)$ and F is the spectral distribution matrix of $X(t)$.

Proof. From the Schwarz inequality,

$$|dF_{jk}(\lambda)|^2 \le dF_{jj}(\lambda)dF_{kk}(\lambda).$$

Hence, the continuity of the F_{jj}, F_{kk} at arbitrary λ gives the continuity of the cross-spectral distribution function F_{jk}.

Therefore, from the propositions 2.1 and 3.1 it follows (4.1). ■

Similarly, we can introduce the aliasing error matrix (as the multidimensional generalization of the aliasing error in scalar process case). Namely the aliasing error matrix $\mathfrak{U}_X(t) = (a_{jk}(t))_{q\times q}$ consists from the so-called cross-aliasing errors $a_{jk}(t) \triangleq E(X_j(t) - X_a^j(t))(X_k(t) - X_a^k(t))^*$. Naturally, the Schwarz inequality gives

$$\begin{aligned}|a_{jk}(t)|^2 &\le a_j(t)a_k(t)\\ &\le 16(\sigma_j^2 - F_{jj}(\omega) + F_{jj}(-\omega))(\sigma_k^2 - F_{kk}(\omega) + F_k(-\omega)).\end{aligned}$$

Here σ_j^2 denotes the variance of the process $X_j(t)$.

V. Summary.

The SCSE $X_a(t)$ of a weakly stationary non-band-limited stochastic process $X(t)$ possesses a spectral representation given by $X_a(t) = \int_{\mathbf{R}}(e^{it\lambda})_{2\omega}dZ(\lambda)$ iff the spectral distribution function $F(\lambda)$ of the process $X(t)$ is continuous at the points $(2k+1)\omega$, $k \in \mathbb{Z}$, (Proposition 2.1). The autocovariance function $K_a(t)$ of such a SCSE $X_a(t)$ has also the spectral representation $\int_{\mathbf{R}}(e^{it\lambda})_{2\omega}dF(\lambda)$. (Proposition 2.2).

Let $X(t)$, $Y(t)$ be non-band-limited stationarily correlated processes with the continuity property and $X_a(t)$, $Y_a(t)$ are its SCSE's to the given bandwidths ω_x, ω_y respectively. Then $X_a(t)$ and $Y_a(t)$ are stationarily correlated iff $\omega_x = \omega_y$. The cross-correlation function $K_a(t)$ of the considered SCSE's is then spectrally represented in the form $\int_{\mathbf{R}}(e^{it\lambda})_{2\omega}dF_{xy}(\lambda)$ where ω is the common bandwidth, (Proposition 3.1; Consequence 3.1).

In the section IV the generalization of foregoing results to the q-variate non-band-limited weakly stationary stochastic processes are given. The aliasing error matrix is introduced (as the multivariate extension of the aliasing error) with the aid of the cross-aliasing error.

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