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# ON SOME QUESTIONS IN QUASI-UNIFORM TOPOLOGICAL SPACES

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Abstract

*Abstract. Partial solution is given here respect to one open problem posed by P. Fletcher and W.F. Lindgren in their monography "Quasi-Uniform spaces".*

*Subject classification: 54 E 15*

1. Introduction.

Since Pervin [3] gave a proof of the fact that every topological space is quasi-uniformizable, there has been considerable interest in the study of quasi-uniform spaces.

If $(X, U)$ is a quasi-uniform space, we obtain a topology $£(U)$ on $X$ by taking as a base for the neighborhood system at $x \in X$, the collection $\{ U(x) : U \in U\}$ and we say that $U$ generates $£(U)$. If $£$ is a topology on $X$ and $£(U)= £$, then $£$ is said to be compatible with $U$. For any topological space $(X, £)$, the collection $\{S_G = G \times G \cup (X - G) \times X : G \in £\}$ is a subbase for a quasi-uniformity $U_p$ (Pervin's quasi-uniformity), which is compatible with $£$.

A topological space satisfies the separation property of being $R_o$ whenever every open set contains the closure $cl(x)$ of each of its points.

For a topological space $X$, $C^*(X)$ will denote the usual ring of bounded continuous real functions.

Each quasi-uniformity $U$ on $X$ induces a conjugate quasi-uniformity $U^{-1}$ on $X$ defined by $U^{-1} = \{ U^{-1} : U \in U\}$. We represent by $U^*$ the uniformity of subbase $U \cup U^{-1}$, such that $\mathcal{L}^* = \mathcal{L}(U^*)$ is the supremus topology $\mathcal{L}(U) \vee \mathcal{L}(U^{-1})$. By $\cap U$ we mean $\cap\{ U : U \in U\}$.

2. Partial solution of Problem A, for Pervin's quasi-uniformity.

In [1], pag. 9, the authors propose the following open problem:

**Problem A.** For which quasi-uniform spaces $(X, U)$ is the family of $\mathcal{L}(U^* \times U^*)$-neighborhoods of $\cap U$ a quasi-uniformity compatible with the topology $\mathcal{L}(U)$?

In the following we contribute to the solution of this problem by determining the conditions under which it is satisfied by Pervin's quasi-uniform space $(X, U_p)$.

**2.1. Definition.** A topological space $(X, \mathcal{L})$ is Locally Minimal whenever every point admits a smallest neighborhood, i.e., $\forall x \in X$, $G_x = \cap\{G \in \mathcal{L} : x \in G\}$ is an open set.

**2.2. Definition.** A topological space is Super-Zero-Dimensional if and only if its topology consists exclusively of clopen subsets.

The proofs of next three lemmas are quite simple and straightforward.

**2.3. Lemma.** For the class of $R_0$ spaces the concepts locally minimal and super-zero-dimensional are equivalent.

**2.4. Lemma.** $(X, U_p)$ is a uniform space if and only if $X$ is super- zero-dimensional.

**2.5. Lemma.** If $\mathcal{L}$ is a topology on $X$ compatible with the quasi- uniformity $U$, then $\cap U = U\{cl\{x\} \times \{x\} : x \in X\}$. Moreover, if $X$ is a $R_0$ space then $\cap U = U\{cl\{x\} \times cl\{x\} : x \in X\}$.

It is well known that the sets $U_{f,r} = \{(x,y) \in X^2 : |f(x) - f(y)| < r\}$, $f \in C^*(X)$, $r > 0$, form a base for a certain uniform structure $U(C^*)$ not necessarily compatible with the topological space $X$. The following result is proved to be useful in our dissertation.

**2.6. Proposition.** Consider the set $\mathbf{R}$ of real numbers with its usual metric uniformity. A function $f$ is quasi-uniformly continuous from $(X, U_p)$ to $\mathbf{R}$ if and only if $f \in C^*(X)$.

*Proof.* To show its necessity, take a quasi-uniformly continuous function $f$ from $(X, U_p)$ to $\mathbf{R}$; by construction of $U_p$ there exist some open sets $G_1, G_2, \ldots, G_n$ such that $|f(x) - f(y)| < 1$ for all $(x,y) \in \cap\{S_{G_i};\ 1 \leq i \leq n\} = U$. Then, for every $x \in X$, we have $f(U(x)) \subset ]f(x) - 1, f(x) + 1[$. Since the family $\{U(x) : x \in X\}$ is a subfamily of all the intersections of the sets $X, G_1, G_2, \ldots, G_n$, we may assume the existence of some points $x_1, x_2, \ldots, x_p$ in $X$ such that $X = \cup\{U(x_j)\ :\ 1 \leq j \leq p\}$. Now, since each $f(U(x_j))$ is bounded in $\mathbf{R}$, so is $f(X)$.

For the sufficiency, let $f \in C^*(X)$, and $k > 0$ such that $f(X) \subset [-k, k]$. Given $r > 0$, let $t_1, t_2, \ldots, t_n \in [-k, k]$ such that the real open intervals with center in $t_i$, $1 \leq i \leq n$, and radius $r/2$ cover $[-k, k]$. For each $i \in \{1, 2, \ldots, n\}$, we consider the open set $G_i = f^{-1}(]t_i - \frac{r}{2}, t_i + \frac{r}{2}[)$. For every $(x,y) \in \cap\{S_{G_i} :\ 1 \leq i \leq n\}$, there is $i \in \{1, 2, \ldots, n\}$ such that $x \in G_i$, thus $y \in G_i$, and $|f(x) - f(y)| \leq |f(x) - t_i| + |t_i - f(y)| < r$. Therefore, $f$ is quasi-uniformly continuous.

**2.7. Proposition.** If $X$ is a $R_0$ topological space, then the following properties are equivalent:

(i) $(X, U_p)$ satisfies Problem A.

(ii) $X$ is super-zero-dimensional.

(iii) $U_p$ is a uniformity.

(iv) $U_p = U(C^*)$.

(v) $X$ is locally minimal.

*Proof.* By means of lemmas 2.3 and 2.4 we need only to show that: (i) → (ii) → (iv) and (ii) → (i).

(i) → (ii). If $(X, U_p)$ satisfies Problem A, we know that the family $U_0$ of all $\pounds(U_p^* \times U_p^*)$-neighborhoods of $\cap U_p$ is a quasi-uniformity compatible with the topology of $X$. For every $(y,z) \in \cap U_p$, since $X$ is $R_0$, there is $x \in X$ such that $(y,z) \in cl(x) \times cl(x)$. If $A = X - cl(y)$, $B = X - cl(z)$, then $(S_A \cap S_A^{-1})(y) = cl(y) \subset cl(x)$ and $(S_B \cap S_B^{-1})(z) = cl(z) \subset cl(x)$, thus $\cap U_p$ is a $\pounds(U_p^* \times U_p^*)$-open set. That is, $\cap U_p \in U_0$. Now, if $F$ is any closed subset of $X$, since $\forall x \in F$, $(\cap U_p)(x) = cl(x) \subset F$, we have that $F$ is a $\pounds(U_0)$-open set, but $\pounds(U_0)$ is the topology of $X$.

(ii) → (iv). From Proposition 2.6, we have obviously $U(C^*) \subset U_p$. On the other hand, if $G$ is any open set, since it is clopen, its characteristic function $g$ is continuous, so $g \in C^*(X)$. But, if $0 < r < 1$, then $U_{g,r} = (G \times G) \cup (X - G \times X - G) \subset S_G$. Thus, $S_G \in U(C^*)$.

(ii) → (i). Let $U_0 = \{U \subset X \times X \ : \ \cap U_p \subset U\}$. Obviously, $U_0$ is a quasi-uniformity in $X \times X$. Now, since $U_p \subset U_0$, and $\forall x \in X$, $(\cap U_p)(x) = cl(x)$ is an open set of $X$, we have that $U_0$ is compatible with the topology of $X$. Finally, since $\cap U_p = U\{cl\{x\} \times cl\{x\} \ : \ x \in X\}$, then $\cap U_p$ is $\pounds(U_p^* \times U_p^*)$-open and therefore $U_0$ coincides with the family of all $\pounds(U_p^* \times U_p^*)$ neighborhoods of $\cap U_p$.

Under the conditions of Proposition 2.7 it is interesting to notice that $U_p$, $U(C*)$ and $U_0$ are all uniformities compatible with $X$ such that, in general, $U_p = U(C*) \subsetneq U_0$ (in order to justify the strict inclusion, just consider $X$ as an infinite discrete space).

Also, the hypothesis of being $R_0$ is proved to be necessary, since, for example, Alexandroff's Connected Dyad ($X = \{0,1\}$, with $\{0\}$ as the only proper open set) satisfies Problem A respect to Pervin's quasi-uniformity $U_p$ and, nevertheless, it is not super-zero-dimensional.

3. Partial solution of Problem A for Locally Minimal spaces.

In the following, we will prove, that Pervin's quasi-uniformity $U_p$, and the FINE quasi-uniformity, satisfy Problem A in the Locally Minimal spaces.

**3.1. Lemma.** Let $(X, \mathcal{L})$ be a Locally Minimal space. Then, the set $B = \cap U_B = U(\{x\} \times G_x \ : \ x \in X)$, where $G_x$ is the smallest neighborhood of $x$, $x \in X$, is a base for the FINE quasi- uniformity $U_B$ on $X$.

With this terminology we have the next proposition.

**3.2. Proposition.** Let $(X, \mathcal{L})$ be a Locally Minimal space. Given $x \in X$, we call $A_x$ the set of points of $X$ which have the open set $G_x$ as its smallest neighborhood. Then: $G_x \cap cl\{x\} = A_x$.

*Proof.* Let $a \in G_x \cap cl\{x\}$, and suppose there is a neighborhood $G_a$ of $a$ such that $a \in G_a \subsetneq G_x$. Then $x$ does not belong to $G_a$ (otherwise, $G_x$ would not be the smallest neighborhood of $x$); Thus $G_a \cap \{x\} = \emptyset$ and therefore $a \notin cl\{x\}$ which is a contradiction.

The other inclusion is obvious.

**3.3. Proposition.** Let $(X, \mathcal{L})$ a Locally Minimal space. Then $U_p$ and $U_B$ satisfy Problem A.

*Proof.* Since the family of $\mathcal{L}$-closed subsets of $X$ is a base for the topology $\mathcal{L}(U_p^{-1})$, ([2], Prop. 2), then $\mathcal{L}(U_p^{-1})$ is a Locally Minimal space and $cl_{\mathcal{L}}\{x\}$ is the smallest $\mathcal{L}(U_p^{-1})$ open neighborhood of $x$, $x \in X$.

On the other hand, since $B^{-1} = U(\{x\} \times cl_{\mathcal{L}}\{x\} \ : \ x \in X\}$ is a base for the quasi-uniformity $U_B^{-1}$, then $cl_{\mathcal{L}}\{x\}$ is also the smallest $\mathcal{L}(U_B^{-1})$ open neighborhood of $x$, $x \in X$. So, $\mathcal{L}(U_p^{-1}) = \mathcal{L}(U_B^{-1})$.

We conclude from Prop. 3.2, that $A_x \in \mathcal{L}^*$, in the above both cases, being $\mathcal{L}* =$

$\mathcal{L}(U_p^*) = \mathcal{L}(U_B^*)$.

From Lemma 3.1, it is easy to observe that $B = U\{A_x \times G_x \,:\, x \in X\}$ and therefore $B$ is a $\mathcal{L}^* \times \mathcal{L}^*$ open subset of the topological product space $X^2$, and in consequence, the family $U_0$ of $\mathcal{L}^* \times \mathcal{L}^*$ neighborhoods of $\cap U_p (= \cap U_B = B)$ agrees with $U_B$, and it is obvious that $\mathcal{L}(U_0) = \mathcal{L}(U_B) = \mathcal{L}$.

References.

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