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PREBOOLEAN MV-ALGEBRAS AS BIPARTITE MV-ALGEBRAS

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ABSTRACT

In this note we characterize bipartite MV-algebras by introducing the notion of preboolean MV-algebras.

1. Introduction.

MV-algebras were introduced by C.C. Chang [1] in 1958 in order to give an algebraic proof of the completness theorem of infinite valued logic of Lukasiewicz-Tarski. Roughly, MV-algebras are a certain generalization of Boolean algebras because the elements in an MV-algebra are not, in general, idempotent.

A. Di Nola, F. Liguori, S. Sessa in [4] introduced the bipartite MV-algebras as MV-algebras A such that $A = M \cup \overline{M}$ for some maximal ideal. In the attempt to generalize the class of bipartite MV-algebras, we considered those MV-algebras A such that $A = \bigcup_{M \in \mathrm{Max}\, A} (M \cup \overline{M})$ where Max A is the set of all maximal ideals of A. We called such algebras preboolean MV-algebras. Substantially these algebras are such that the order of every element of its opposite is infinite. In this brief note we prove that the class of the preboolean MV-algebras and the class of the bipartite MV-algebras coincide. That is, if $A = \bigcup_{M \in \mathrm{Max}\, A} (M \cup \overline{M})$ then it is possible to individue one maximal ideal M such that $A = M \cup \overline{M}$.

2. Preliminaries.

We recall the axioms of MV-algebra and some relative definitions. For a deeper knowledge of this structure we remind to [1], [2], [3], [5], [6], [7].

An MV-algebra is an algebraic structure $(A, +, \cdot, -, 0, 1)$ where A is a non-empty set, $+$ and $\cdot$ are two binary operations, $-$ is a unary operation, 0 and 1 are constant elements of A such that:

1) $(A, +, 0)$ and $(A, \cdot, 1)$ are commutative semigroups with identity.
2) $x + \overline{x} = 1, \quad x \cdot \overline{x} = 0, \quad \overline{0} = 1$ for all $x \in A$.
3) $\overline{x + y} = \overline{x} \cdot \overline{y}, \quad \overline{x \cdot y} = \overline{x} + \overline{y}, \quad \overline{\overline{x}} = x$ for all $x, y \in A$.
4) Defining $\vee$ and $\wedge$ by $x \vee y = x + \overline{x}y$, $x \wedge y = x(\overline{x} + y)$ we have that $(A, \vee, 0)$, $(A, \wedge, 1)$ are to be commutative semigroups with identity.
5) $x(y \vee z) = xy \vee xz, \; x + (y \wedge z) = (x + y) \wedge (x + z)$ for all $x, y, z \in A$.

From these axioms it follows that the structure $(A, \vee, \wedge, 0, 1)$ is a distributive lattice with least element 0 and greatest element 1.

In the sequel we will agree that $0x = 0$, $(n + 1)x = nx + x$ and $x^0 = 1$.

Definition 1.

The order of an element $x \in A - \{0\}$, in simbols ord(x), is the least integer m such that $mx = 1$. If no such integer m exists then ord(x)=∞.

We agree to say that ord0=∞.

Definition 2.

An ideal of A is a non-empty subset $I \subseteq A$ such that

i) $x, y \in I \Rightarrow x + y \in I$.

ii) $x \in I$, $y \leq x \Rightarrow y \in I$.

Definition 3.

An ideal M is a maximal ideal of A, if M is a proper ideal and whenever I is an ideal such that $M \subseteq I \subseteq A$, then either $I = M$ or $I = A$.

By Th. 4.6 of [1], the set Max A of all maximal ideals of A is non-empty.

3. Preboolean MV-algebras

Definition 4.

An MV-algebra A is called preboolean if $\text{ord}(x) = \infty$ or $\text{ord}(\overline{x}) = \infty$ for every $x \in A$, or equivalently if $\text{ord}(x \wedge \overline{x}) = \infty$ for every $x \in A$.

Let I be a proper ideal of an MV-algebra A. The subalgebra A_I generated by I, is equal to $I \cup \overline{I}$ where $\overline{I} = \{x \in A / \ \overline{x} \in I\}$.

Theorem 1.

The following testaments are equivalent:

1) A is a preboolean MV-algebra.

2) For every $x \in A$, there is $M \in \text{Max}\, A$ such that $x \in M$ or $\overline{x} \in M$

3) $A = \bigcup\limits_{M \in \text{Max}\, A} A_M$

Proof.

1) $\Longleftrightarrow$ 2) it is obvious.

Now we prove that $1 \Longleftrightarrow 3$. Let A be preboolean and $x \in A$. By 2) there is $M \in \text{Max}\, A$ such that $x \in M$ or $\overline{x} \in M$, hence $x \in A_M$.

So $A \subseteq \bigcup\limits_{M \in \text{Max}\, A} A_M \subseteq A$ and $A = \bigcup\limits_{M \in \text{Max}\, A} A_M$.

Viceversa if $A = \bigcup\limits_{M \in \text{Max}\, A} A_M$, then for each $x \in A$, $x \in A_M$ for some $M \in \text{Max}\, A$. If $x \in M$, then $\text{ord}(x) = \infty$, if $x \notin M$, then $\overline{x} \in M$ and $\text{ord}(\overline{x}) = \infty$.

Corollary 1.

If A is a preboolean MV-algebra, then for each $x \in A$ there is $M \in \operatorname{Max} A$ such that $\frac{x}{M} \in \{0,1\}$.

Proof. Obvious by Theorem 1.

In order to prove that every preboolean MV-algebra is a bipartite MV-algebra, we premise the following:

Lemma 1.

If A is a preboolean MV-algebra, then for every finite family $\{x_1, x_2, \ldots, x_n\} \subseteq A$ there is a maximal ideal M such that $\{x_1 \wedge \overline{x}_1, x_2 \wedge \overline{x}_2, \ldots, x_n \wedge \overline{x}_n\} \subseteq M$.

Proof.

Ab absurdo, let $\{x_1, \ldots, x_n\} \subseteq A$ be a finite family such that for every maximal ideal M there exists $i_M \in \{1, 2, \ldots, n\}$ such that $x_{i_M} \wedge \overline{x}_{i_M} \notin M$. Set $I_0 = \{i_M | M \in \operatorname{Max} A\}$ and $t = \bigvee_{i_M \in I_0} (x_{i_M} \wedge \overline{x}_{i_M}) \in A$. For a fixed $M \in \operatorname{Max} A$ consider $\frac{t}{M} = \bigvee_{i_M \in I_0} \left(\frac{x_{i_M} \wedge \overline{x}_{i_M}}{M} \right)$. Since $x_{i_M} \wedge \overline{x}_{i_M} \notin M$, $\frac{x_{i_M} \wedge \overline{x}_{i_M}}{M} \neq 0$, hence $\frac{t}{M} \neq 0$. To prove that $\frac{t}{M} \neq 1$, we note that $\{x_{i_M} \wedge \overline{x}_{i_M}\}_{i_M \in I_0}$ is a finite and linearly ordered family, so $\frac{t}{M} = \frac{x_{i_{M_0}} \wedge \overline{x}_{i_{M_0}}}{M}$ for some $i_{M_0} \in I_0$.

If $\frac{x_{i_{M_0}} \wedge \overline{x}_{i_{M_0}}}{M} = 1$, then $x_{i_{M_0}} \vee \overline{x}_{i_{M_0}} \in M$ that is absurd.

By Corollary 1 thesis follows.

We recall that Inf $A = \{x \in A \ / \ \exists y \in A \text{ and } x = y \wedge \overline{y}\}$ and <Inf A > is the ideal generated by Inf A.

By Th. 4.11 of [4] we know that A is bipartite iff <Inf A >$\neq A$.

Finally we prove:

Theorem 2.

If A is a preboolean MV-algebra, then A is a bipartite MV-algebra.

Proof.

Suppose that A is not bipartite, then by Th. 4.11 of [4] $<\mathrm{Inf}\ A> = A$. So there are $x_1, x_2, \ldots, x_n \in A$ such that $1 = (x_1 \wedge \overline{x}_1) + (x_2 \wedge \overline{x}_2) + \ldots + (x_n \wedge \overline{x}_n)$. By Lemma 1 there is a maximal ideal M containing $x_1 \wedge \overline{x}_1$, $x_2 \wedge \overline{x}_2, \ldots, x_n \wedge \overline{x}_n$ that is absurd.

Observing that every bipartite MV-algebra is a preboolean MV-algebra, we remark that the attempt to generalize the class of bipartite MV-algebras produced a characterization of bipartite MV-algebras. That is

Theorem 3.

A is a preboolean MV-algebra iff A is a bipartite MV-algebra.

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