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THE INTERNAL RATE OF RETURN OF FUZZY CASH FLOWS

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ABSTRACT

An internal rate of return (IRR) of an investment or financing project with cash flow $(a_0, a_1, a_2, \ldots, a_n)$ *is usually defined as a rate of interest* r *such that*

$$a_0 + a_1(1+r)^{-1} + \ldots + a_n(1+r)^{-n} = 0.$$

If the cash flow has one sign change then the previous equation has a unique solution $r > -1$.

Generally the IRR technique does not extend to fuzzy cash flows, as it can be seen with examples (see [2]). In this paper we show that under suitable hypothesis a unique fuzzy IRR exists for a fuzzy cash flow.

Keywords: *Mathematics of finance, Fuzzy numbers.*

Introduction.

In [2] Buckley has proposed the fuzzy extension of the mathematics of finance. His paper develops fuzzy analogues of the elementary compound interest problems, as the future value, the present value and the internal rate of return (IRR) of a cash flow.

With respect to the last problem he has proved the possibility of non existence of the IRR of a fuzzy cash flow where the fuzzy numbers involved, $A_0, A_1, \ldots, A_n$, have one sign change (in the crisp case this is the most important cash flow situation and it is well-known that a unique IRR is guaranteed).

This situation is not an exception when passing from classical equation to the fuzzy analogues, but the rule, as one can see with very simple equations (see Examples 1 and 3 of [1]).

In this note we indicate some quite natural hipotheses on the fuzzy numbers, $A_0, A_1, \ldots, A_n$, in order to a unique fuzzy IRR exists, at least in the previous simplest situation (one only sign change).

Observe that the class of fuzzy numbers we consider is more general than the one considered by Buckley: for example the fuzzy numbers we consider can be not continuous.

The technique we use is related to the procedure of [1]. Indeed, in spite of its very specific character, this note supports a rather broader point: the fuzzy arithmetic is semplified if the membership function is inverted and represented as a pair of functions, the functions being the boundaries of the fuzzy number's cut set.

1. Definitions and some basic result.

A *fuzzy number* (f.n.) A is defined by means of its membership function $\mu_A : \mathbf{R} \to [0,1]$.

The y-cut of A is defined by: $C_A^y = \{x \in \mathbf{R} \ / \ \mu_A(x) \geq y\}$.

The *support* of A is the set $\text{supp}A = \{x \in \mathbf{R} : \ \mu_A(x) > 0\}$.

We say that A is *convex* if the y-cuts of A are convex for $0 < y \leq 1$.

A is called *bounded* if supp$A$ is bounded, *positive* if $\text{supp}A \subset]0, +\infty[$, *negative* if $\text{supp}A \subset]-\infty, 0[$, *normal* if there exists x_0 such that $\mu_A(x_0) = 1$.

Moreover we say that A is *upper semicontinuous* (u.s.c.) if its membership function is upper semicontinuous.

If A is an u.s.c. normal, convex and bounded f.n. then for every $y \in]0,1]$ the y-cut of A is a closed, bounded interval $[f_1(y/A), f_2(y/A)] \neq \emptyset$; we shall suppose all fuzzy numbers considered are normal, and we set: $(\text{supp}A)^- = [f_1(0/A), f_2(0/A)]$.

We call $f_1(y/A)$ and $f_2(y/A)$ the *cut-functions* of A. Of course $f_1(y/A)(f_2(y/A))$ is an (not necessarily strictly) increasing (decreasing resp.) function of $y \in [0,1]$.

Remark. Observe that in [2] Buckley defines the fuzzy number A by means of two continuous and invertible functions $f_1(y/A)$ and $f_2(y/A)$ defined in the interval [0,1]. Since we consider convex, bounded and u.s.c. f.n. A, the cut-functions $f_1(y/A)$ and $f_2(y/A)$ only need to be left-continuous. Indeed we have

$$C_A^{y_0} = [f_1(y_0/A), f_2(y_0/A)] = \bigcap_{y<y_0} C_A^y = \bigcap_{y<y_0} [f_1(y/A), f_2(y/A)] =$$
$$= [\sup_{y<y_0} f_1(y/A), \inf_{y<y_0} f_2(y/A)]$$

hence

$$f_1(y_0/A) = \sup_{y<y_0} f_1(y/A) = \lim_{y\to y_0^-} f_1(y/A), \quad f_2(y_0/A) = \inf_{y<y_0} f_2(y/A) = \lim_{y\to y_0^-} f_2(y/A).$$

A *flat fuzzy number* A is a f.n. whose membership function is given by:

$$\mu_A/x(= \begin{cases} 0 & \text{if} \quad x < a \\ (x-a)/u & \text{if} \quad a \le x \le a+u \\ 1 & \text{if} \quad a+u \le x \le b-v \\ (b-x)/v & \text{if} \quad b-v \le x \le b \\ 0 & \text{if} \quad x > b. \end{cases}$$

where $a \le a+u \le b-v \le b$. In particular if $a+u = b-v$ we have *triangular* f.n.; obviously the y-cut of A is the set $\{x \in \mathbf{R} \; / \; \mu_A(x) \ge y\} = [a+yu, b-yv]$ for $0 < y \le 1$.

Observe that if $h : \mathbf{R}^n \to \mathbf{R}$ is a function and if $A_1, \ldots, A_n$ are f.n., then by means of Zadeh's extension principle, we can obtain a f.n. A whose membership function is

$$\mu_A(z) = \vee\{\mu_A(x_1) \wedge \ldots \wedge \mu_A(x_n) \; / \; h(x_1, x_2, \ldots, x_n) = z\}.$$

In particular if $*$ is a binary operation in $\mathbf{R}$ the previous procedure gives its extension to the set of the fuzzy numbers.

Proposition 1. If $A_1, \ldots, A_n$ are normal then A is normal; if h is continuous and $A_1, \ldots, A_n$ are convex then A is convex (see [1], Proposition 2.2).

Proposition 2. If $*$ is continuous and A and B are bounded u.s.c. f.n. then

$$C_A^y * C_B^y = C_{A*B}^y,$$

$C_A^y * C_B^y$ being the set $\{x_1 * x_2 \ / \ x_1 \in C_A^y, \ x_2 \in C_B^y\}$. (See [1]), Proposition 2.4). In particular $A * B$ is u.s.c.

2. Fuzzy cash flows and the internal rate of return.

Consider an investment or financing project with cash flow $(a_0, a_1, \ldots, a_n)$. An internal rate of return (IRR) is usually defined as a rate of interest r such that:

$$a_0 + a_1(1+r)^{-1} + \ldots + a_n(1+r)^{-n} = 0.$$

If $a_0 < 0$ and if the cash flow has one sign change, then the previous equation has a unique solution $r > -1$.

The simplest and most important cash flow situation is when $a_k > 0$ for $1 \leq k \leq n$, i.e., the case of a pure investment project.

In this Section we consider a fuzzy cash flow of a pure investment project $(-A_0, A_1, \ldots, A_n)$ with A_k positive u.s.c. convex and bounded f.n. for every $k \in \{0, 1, 2, \ldots, n\}$.

Definition. A *fuzzy internal rate of return* (f.IRR) is an u.s.c., convex and bounded fuzzy number $R > -1$ (that is $f_1(y/A_k) > -1$ for every $y \in [0,1]$) satisfying the fuzzy equation:

$$A_0 = \sum_{k=1}^{n} A_k \otimes (1 \oplus R)^{-k}. \tag{1}$$

where $\otimes$, $\oplus$ denote the multiplication and the addition extended to fuzzy numbers by means of the Zadeh's extension principle.

If A_k and R are given f.n. convex, bounded and u.s.c., then, by Proposition 1 and 2 of Section 1, A_0 is a convex u.s.c. f.n. whose cut-functions are:

$$f_1(y/A_0) = \sum_{k=1}^{n} f_1(y/A_k)[1 + f_2(y/R)]^{-k} \tag{2}$$

$$f_2(y/A_0) = \sum_{k=1}^{n} f_2(y/A_k)[1 + f_1(y/R)]^{-k} \tag{3}$$

In order to solve Equation (1) we have the following:

Proposition 1. If, for every $k \in \{1, 2, \ldots, n\}$ we have that

$$\begin{cases} f_2(y/A_k)/f_2(y/A_0) & \text{is increasing} \\ f_1(y/A_k)/f_1(y/A_0) & \text{is decreasing} \end{cases} \tag{4}$$

in the interval [0,1] and

$$f_2(1/A_k)/f_1(1/A_k) \leq f_2(1/A_0)/f_1(1/A_0) \tag{5}$$

then there exists a unique f.n. $R > -1$ convex and u.s.c. satisfying Equation (1).

Proof. Consider the equation (2) and (3) in the unknowns $f_1(y/R)$, $f_2(y/R)$. It is well-known that they admit one solution greater than -1, respectively, $f_2(y/R) > -1$ and $f_1(y/R) > -1$.

We shall prove that

i) $f_1(1/R) \leq f_2(1/R)$

ii) $f_1(y/R)$ is increasing function of y, $f_2(y/R)$ is decreasing.

Proof of i). If $f_1(1/R) > f_2(1/R)$, then we have,

$$\forall k \in \{1, 2, \ldots, n\} \qquad [1 + f_1(y/R)]^{-k} < [1 + f_2(y/R)]^{-k}$$

and hence, by (5), (2), (3):

$$f_2(1/A_0) = \sum_{k=1}^{n} f_2(1/A_k)[1 + f_1(1/R)]^{-k} <$$
$$< f_2(1/A_0)/f_1(1/A_0) \sum_{k=1}^{n} f_1(1/A_k)[1 + f_2(1/R)]^{-k} =$$
$$= f_2(1/A_0),$$

an absurdity. Thus $f_1(1/R) \leq f_2(1/R)$.

Proof of ii). We have to prove that

$$0 \leq y_1 < y_2 \leq 1 \Rightarrow f_1(y_1/R) \leq f_1(y_2/R). \tag{6}$$

If there exist y_1, y_2 such that $0 \leq y_1 < y_2 \leq 1$ and $f_1(y_1/R) > f_1(y_2/R)$, then, by (2), (3), (5) and the first assertion in (4), we obtain:

$$1 = \sum_{k=1}^{n} [1 + f_1(y_1/R)]^{-k} f_2(y_1/A_k)/f_2(y_1/A_0) <$$
$$< \sum_{k=1}^{n} [1 + f_1(y_2/R)]^{-k} f_2(y_2/A_k) f_2(y_2/A_0) = 1,$$

an absurdity. Thus implication (6) holds for every y_1, y_2.

Analogously one proves that $f_2(y/R)$ is decreasing.

Consider the intervals $[f_1(y/R), f_2(y/R)]$, $y \in [0,1]$. As is well known, they constitute the family of the cuts of a fuzzy number if: for every $y_0 \in]0,1]$,

$$[f_1(y_0/R), f_2(y_0/R)] = \bigcap_{y<y_0} [f_1(y/R), f_2(y/R)]$$

and this is true if

$$\lim_{y \to y_0^-} f_1(y/R) = \sup_{y<y_0} f_1(y/R) = f_1(y_0/R), \qquad \lim_{y \to y_0^-} f_2(y/R) = \inf_{y<y_0} f_2(y/R) = f_2(y_0/R).$$

Now, these equalities are an immediate consequence of Equations (2) and (3) and the left-continuity of cut-functions of $A_0, A_1, \ldots, A_n$.

Thus we can consider the fuzzy number R whose membership function is defined by:

$$\mu_R(x) = \sup y \cdot \chi_{[f_1(y/R), f_2(y/R)]}(x)$$

and R is convex, bounded, normal and u.s.c. since

$$\{x : \mu_R(x) \geq y\} = [f_1(y/R), f_2(y/)].$$

It is now obvious that R is solution of Equation (1).

Remark 1. Notice that by (5) we have in particular, that if $f_2(1/A_0) = f_1(1/A_0)$, that is the peak of A_0 is constitued by a unique point, also $f_2(1/A_k) = f_1(1/A_k)$, $\forall k \in \{1, 2, \ldots, n\}$, that is the peaks of $A_1, A_2, \ldots, A_n$ are constitued by a unique point.

In particular we have

Proposition 2. Let $A_0, A_1, \ldots, A_n$ be positive flat fuzzy numbers, that is $f_1(y/Ak) = a_k + yu_k$,

$f_2(y/A_k) = b_k - yv_k$, $a_k \leq a_k + u_k \leq b_k - v_k \leq b_k$, for every $k \in \{0, 1, 2, \ldots n\}$. If

$$u_k/a_k \leq u_0/a_0, \qquad v_k/b_k \leq v_0/b_0 \tag{7}$$

$$(b_k - v_k)/(a_k + u_k) \leq (b_0 - v_0)/(a_0 + u_0) \tag{8}$$

then there exists a unique convex, u.s.c. f.n. $R > -1$ satisfying Equation (1).

Proof. Indeed (7) entails that

$$(b_k - yv_k)/(b_0 - yv_0) = f_2(y/A_k)/f_2(y/A_0)$$

is increasing and

$$(a_k + yu_k)/(a_0 + yu_0) = f_1(y/A_k)/f_1(y/A_0)$$

is decreasing.

(8) is nothing but (5).

Remark 2. If $(b_0 - v_0)/(a_0 + u_0) = 1$, that is A_0 is triangular, then (8) entails th $A_1, A_2, \ldots, A_n$ are all triangular numbers. And if $u_0 = v_0 = 0$, that is A_0 is crisp, then follows from (7) that also $u_k = v_k = 0$, $\forall k \in \{1, 2, \ldots, n\}$, that is $A_1, A_2, \ldots, A_n$ are cris too.

Example 1. Let A_0, A_1, A_2 be triangular numbers with parameters

$$u_0 = v_0 = 10, \qquad v_k = v_k = 5, \qquad a_0 + u_0 = b_0 - v_0 = 110, \qquad a_k + u_k = b_k - v_k = 7($$

(7) and (8) are verified. Equation (2) and (3) become:

$$\sum_{k=1}^{n}(65 + y5)[1 + f_2(y/R)]^{-k} = 100 + 10y.$$
$$\sum_{k=1}^{n}(75 - y5)[1 + f_1(y/R)]^{-k} = 120 - 10y.$$

Hence

$$f_1(y/R) = 1/(48 - 4y)[-33 + 3y + \surd(9y^2 - 246y + 1665)]$$
$$f_2(y/R) = 1/(40 - 4y)[-27 + 3y + \surd(9y^2 - 210y + 1209)] \tag{9}$$

and $f_1(1/R) = f_2(1/R) \cong 0,1770$ that is the IRR in the crisp case.

Notice that $f_1(0/R) \cong 0,1625$ and$f_2(0/R) \cong 0.1942$ hence the IRR is not less that 0,1625 and not greatest that 0,1942.

Example 2. Let A_0, A_1, A_2 be flat numbers with parameteres

$$a_0 = 100 \qquad u_0 = v_0 = 10 \qquad b_0 = 140$$
$$a_k = 65 \qquad u_k = v_k = 5 \qquad b_k = 85$$

(7) and (8) are satisfied. Equations (2) and (3) become:

$$\sum_{k=1}^{n}(65 + y5)[1 + f_2(y/R)]^{-k} = 100 + 10y$$
$$\sum_{k=1}^{n}(85 - y5)[1 + f_1(y/R)]^{-k} = 140 - 10y$$

Hence we obtain that $f_2(y/R)$ is given by (9) and

$$f_1(y/R) = 1/(56 - 4y)[-39 + 3y + \sqrt{9y^2 - 282y + 2193}].$$

Observe that

$$f_1(1/R) \cong 0,1503 < f_2(1/R) \cong 0,1770$$
$$f_1(1/R) \cong 0,1398 f_2(1/R) \cong 0,1942.$$

Example 3. Let $A_0, A_1, \ldots, A_n$ defined by

$$f_1(y/A_0) = \begin{cases} 1, & \text{if } y \in [0,1/2] \\ 2, & \text{if } y \in]1/2,1] \end{cases}; \qquad f_2(y/A_0) = 4, \quad \forall y \in [0,1]$$

$$f_1(y/A_1) = \begin{cases} 1/2, & \text{if } y \in [0,1/2] \\ 1, & \text{if } y \in]1/2,1] \end{cases}; \qquad f_2(y/A_1) = 2, \quad \forall y \in [0,1]$$

$f_1(y/A_2) = 3/2$; $f_2(y/A_2) = 5/2$, $\forall y \in [0,1]$.

(4) and (5) are satisfied.

Equations (2) become

$$1 = 1/2(1 + f_2(y/R))^{-1} + 3/2(1 + f_2(y/R))^{-2}, \text{ for } y \in [0,1/2]$$

and

$$2 = (1 + f_2(y/R))^{-1} + 3/2(1 + f_2(y/R))^{-2}, \text{ for } y \in]1/2,0]$$

from which we obtain

$$f_2(y/R) = \begin{cases} 0,5, & \text{if } y \in [0,1/2] \\ 0,1516, & \text{if } y \in]1/2,1]. \end{cases}$$

Equations (3) become:

$$4 = 2(1 + f_1(y/R))^{-1} + 5/2(1 + f_1(y/R))^{-2},$$

that gives

$$f_1(y/R) = 0,0734.$$

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