Notas Breves

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ON FUNCTIONS THAT CANNOT BE MV-TRUTH VALUES IN ALGEBRAIC STRUCTURES

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ABSTRACT

It is shown, in a general frame and playing with idempotency, that in order to have on a given lattice a Multiple Valued Logic preserving the lattice structure, the only t-norms and t-conorms allowing to modelize the truth values of $a \vee b$, $a \wedge b$ and $a \rightarrow b$ are Min and Max, respectively, apart from ordinal sums.

Let $(L, *)$ be an algebraic structure, that is a non-empty set L with a binary operation $L \times L \rightarrow L, *(a,b) = a * b$, and let V be a function $L \rightarrow [0,1]$ "evaluating" the elements of L in some predetermined sense (a certainty factor in the language of Expert Systems) such that, for all $(a,b) \in L \times L$,

$$V(a * b) = T(V(a), V(b)) \tag{1}$$

being T a t-norm [1].

Theorem 1. Any function $V : L \rightarrow [0,1]$, verifying (1), preserves the relation $a \leq b$ given in L by $a * b = a$.

Proof. If $a \leq b$, as $T \leq$Min [1], we have $V(a) = V(a * b) \leq \text{Min}\,(V(a), V(b)) \leq V(b)$.

Theorem 2. If there exists $z \in L$ such that $z = z * z$ and $V(z) \in (0,1)$, then the only t-norms T for which (1) is possible are Min and the ordinal sums.

Proof. As $V(z) = V(z * z) = T(V(z), V(z))$, we see that $V(z)$ is an idempotent for T, so that T has to be [1] either Min or an ordinal sum.

Theorems 1 and 2 can be used when L is an lower-semilattice [2] with $* = \wedge$, the meet operation. In that case, functions $V : L \to [0,1]$ verifying $V(a \wedge b) = T(V(a), V(b))$ and effectively ranging in (0,1) are candidates to be Multiple-Valued Truth Values, but with the strong limitation of T being only Min or an ordinal Sum. The same happens if L is a set of propositions equipped with a sort of conjunction and with a certainty factor V [3], when there exists just one proposition p such that $p \wedge p = p$ and $0 < V(p) < 1$.

Similar proofs are valid for the next theorems.

Theorem 3. If there exists $z \in L$ such that $z * z = z$ and $V(z) \in (0,1)$, then the only t-conorms S for which it holds

$$V(a * b) = S(V(a), V(b)) \tag{2}$$

for all $(a, b) \in L \times L$ are the Max and the ordinal sums.

Theorem 4. Any function $V : L \to [0,1]$, verifying (2) preserves the relation $a \leq b$ given in L by $a * b = b$.

Theorems 3 and 4 can be used when L is an upper-semilattice [2] with $* = \vee$ the join operation. In such case, functions V verifying $V(a \vee b) = S(V(a), V(b))$ and effectively ranging in (0,1) are also candidates to be MV-Truth Values, but with the limitation of S being either Max or an ordinal sum. The same happens if L is a set of propositions equipped with some disjunction $\vee$ and a certainty factor V [3], when there exists just one proposition p such that $p \vee p = p$ and $0 < V(p) < 1$.

Consequently, for lattices $(L, \vee, \wedge)$, the only possibilities for representing actual MV-Truth Values using t-norms or t-conorms by means of (1) or (2) respectively, are $T = Min$ or $S = Max$ and, in both cases, ordinal sums. Other t-norms or t-conorms will give only

$V(a) \in \{0,1\}$ for each $a \in L$, and no actual Multiple-Valued Logic will be available with such functions V. This result completes what is proven in [4], [5] and [6].

In the last case, and provided that $\neg\neg a = a$ for each $a \in L$, it should be pointed out that it is excluded the possibility of giving the truth value of $a \to b = \neg a \vee b$ by means of $V(a \to b) = 1 - T(1 - V(\neg a),\ 1 - V(b))$, where $V(\neg a) = 1 - V(a)$, by virtue of $V(\neg a \to a) = V(a \vee a) = V(a) = 1 - T(V(\neg a), 1 - V(a)) = 1 - T(1 - V(a), 1 - V(a))$, or $1 - V(a) = T(1 - V(a), 1 - V(a))$; and, if $V(a) \in (0,1)$, T should be either Min or an ordinal sum.

Thus, it is clear that for compatibility with a lattice structure on L only T =Min or T = an ordinal sum, are admissible t-norms. Of course, it is possible to use other t-norms to modelize $V(a \to b)$ only, as is the case of Luckasiewicz's Logic [7] in which $V(a \vee b)) = Max(V(a), V(b))$, $V(a \wedge b) = Min(V(a), V(b))$ and $V(a \to b) = Min(1, 1 - V(a) + V(b)) = 1 - L(V(a), 1 - V(b))$, where $L(x,y) = Max(0, x + y - 1)$ is the so-called Luckasiewicz t-norm [1]. This is also the case of Zadeh's Fuzzy Logic [8] with $L = [0,1]^{[0,1]}$ the set of all fuzzy sets over $[0,1]$, and where $(\mu_A \cap \mu_B)(x) = Min(\mu_A(x), \mu_B(x))$, $(\mu_A \cup \mu_B)(x) = Max(\mu_A(x), \mu_B(x))$ and $(\mu_A \to \mu_B)(x) = Min(1, 1 - \mu_A(x) + \mu_B(x))$, for each $x \in [0,1]$, μ_A and μ_B being fuzzy sets over $[0,1]$. In those cases no idempotency "pathology" arises.

Remark.

(I) If $a \to b = \neg a \vee (a \wedge b)$ and $\neg a \vee a = 1$, for each $a \in L$ (Excluded Middle Principle) then, setting $V(a \to b) = V(\neg a \vee (a \wedge b)) = 1 - T(V(a), 1 - V(b))$, we have $V(\neg a \vee a) = 1 = V(a \to a) = 1 - T(V(a), 1 - V(a))$ and $T(V(a), 1 - V(a)) = 0$ for all $a \in L$. This excludes t-norms "like" T=Min or T=Prod if we want to preserve both the Excluded Middle Principle and the actual MV-Truth Value. For t-norms "like L" (see [9]), $T = \phi^{-1} \circ L \circ (\phi \times \phi)$ where ϕ is an automorphism of $[0,1]$, we will have $\phi^{-1}(L(\phi(V(a)), \phi(1 - V(a))) = 0$, that is $L(\phi(V(a)), \phi(1 - V(a))) = 0$, or $Max(0, \phi(V(a)) + \phi(1 - V(a)) - 1) = 0$. Thus, ϕ must verify $\phi(V(a)) + \phi(1 - V(a)) \leq 1$ for each $a \in L$ (as is actually in the Luckasiewicz case for $\phi = j$, the identity function, so that both the Excluded Middle Principle and the MV-Truth Value [4] hold.

(II) Of course, it is always possible to redefine $V(a \vee b)$ in lattices with least element 0, and for "measure" purposes, by setting $V(a \vee b) = S(V(a), V(b))$ if $a \wedge b = 0$. In the case of orthomodular lattices [2] in which $a \to b = a' \vee (a \wedge b)$, a' being the orthocomplement of a (for Boolean algebras we have $a \to b = a' \vee b$) it is possible to write $V(a \to b) = S(V(a'), V(a \wedge b))$ because $a' \wedge (a \wedge b) = 0$. Nevertheless, in all those cases we do not obtain what is usually known as a truth-value function or truth-table for logical connectives.

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