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# AN EXAMPLE OF SEMILINEAR TOPOLOGIES

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In this paper by a **linear space** always is meant a linear space over the field $\mathbb{R}$ of real numbers and of a positive dimension. (Thus the singleton $\{0\}$ is **not** a linear space).

If $X$, $Y$ are linear spaces, then the product $X \times Y$ also is considered to be a linear space with the algebraic operations defined ("coordinatewise") with the aid of those existing in $X$ and in $Y$. The spaces $X$ and $Y$ may be identified with $X \times \{y\}$ and $\{x\} \times Y$ $(x \in X, y \in Y)$, respectively, so sometimes $X$ and $Y$ will be treated as subsets (subspaces) of $X \times Y$.

Let $E$ be a linear space and assume that $E$ is endowed with a topology $\mathcal{T}$. The topology $\mathcal{T}$ is called **linear** (and $E$ is then called a **linear topological space**) iff the function $\varphi : \mathbb{R} \times E \times E \longrightarrow E$ defined by

$$\varphi(\lambda, a, b) = \lambda a + b \tag{1}$$

is continuous ($\mathbb{R}$ is understood to be endowed with the natural topology of the real line). If function (1) is separately continuous with respect to each variable, then the topology $\mathcal{T}$ is called **semilinear** (cf. [3] and also [1]). Of course, every linear topology is also semilinear. Another example of a semilinear topology is furnished by the core topology (cf. [1]- [4]), which exists in every linear space and is generated by the linear structure of the space.

Let $E$ be a linear space and let $A \subset E$ be an arbitrary subset of $E$. A point $a \in A$ is called **algebraically interior** to $A$ iff for every $h \in E$ there exists an $\epsilon = \epsilon(a, b) > 0$ such that

$$a + \lambda b \in A \quad \text{for} \quad \lambda \in (-\epsilon, \epsilon). \tag{2}$$

The set of points that are algebraically interior to $A$ is denoted core $A$:

(3) $$\text{core } A := \{a \in A\, a \text{ is algebraically interior to } A\},$$

and $A$ is called **algebraically open** whenever $A = \text{core } A$. The family

(4) $$\mathcal{T}(E) := \{A \subset E | A = \text{ core } A^{-}\}$$

of all algebraically open subsets of $E$ is a topology in $E$ and is called the **core topology** in $E$ (cf. [1]-[4]). We have (cf. [2] and also [1]).

**Lemma 1.** Let $E$ be a linear space. If dim $E = 1$, then the core topology $\mathcal{T}(E)$ is linear (coincides with the natural topology of the real line). If dim $E \geq 2$, then $\mathcal{T}(E)$ is semilinear, but not linear.

The question about the existence of other semilinear topologies remained unanswered for some time. Recently Z. Kominek [1] has proved -assuming the axiom of choice and the continuum hypothesis- that if $Z$ is a linear topology Baire and Hausdorff space of dimension greater than one and satisfying the second axiom of countability, then there exists in $Z$ a semilinear topology which is neither linear nor identical with the core topology $\mathcal{T}(Z)$.

The purpose of the present note is to give another example of semilinear topologies with similar properties. In our construction we also use the axiom of choice, but not the continuum hipothesis.

Before stating our main result we introduce some notations. If $E$ endowed with a topology $\mathcal{T}$ is a topological space and $E_0 \subset E$ is a subset of $E$, then the topology in $E_0$ inherited from $(E, \mathcal{T})$ will be denoted $\mathcal{T}|E_0$. If $X$ and $Y$ with the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively, are topological spaces, then the product topology in $X \times Y$ generated by $\mathcal{T}_X$ and $\mathcal{T}_Y$ will be denoted $\mathcal{T}_X \times \mathcal{T}_Y$. Because of the identification mentioned at the beginning of this paper, the symbols $(\mathcal{T}_X \times \mathcal{T}_Y)|X$ and $(\mathcal{T}_X \times \mathcal{T}_Y)|Y$ make sense.

Some relations between various topologies are listed in the following lemmas.

**Lemma 2.** Let $X$, $Y$ (with the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively) be topological spaces, let $X_0 \subset X$ and $Y_0 \subset Y$ be respective subsets, let $E$ be a linear space and let $E_0 \subset E$ be a linear subspace of $E$. Then (cf. [4]);

i) $(\mathcal{T}_X \times \mathcal{T}_Y)|X = \mathcal{T}_X, \quad (\mathcal{T}_X \times \mathcal{T}_Y)|Y = \mathcal{T}_Y.$

ii) $\mathcal{T}(E)|E_0 = \mathcal{T}(E_0)$.

iii) $\mathcal{T}_X|X_0) \times (\mathcal{T}_Y|Y_0) = (\mathcal{T}_X \times \mathcal{T}_Y)|(X_0 \times Y_0)$.

*Proof.* Relations (i)-(iii) are a more or less immediate consequence of the definitions of the notions occurring there. As an example we prove here (ii).

Let $A \in \mathcal{T}(E)|E_0$. This means that there exists a set $\hat{A} \in \mathcal{T}(E)$ such that

$$A = \hat{A} \cap E_0. \tag{5}$$

Take arbitrary $a \in A$ and $b \in E_0$. By (5) we have $a \in \hat{A}$, and clearly $b \in E$. Thus there exists an $\epsilon > 0$ such that

$$a + \lambda b \in \hat{A} \quad \text{for} \quad \lambda \in (-\epsilon, \epsilon), \tag{6}$$

since $\hat{A} \in \mathcal{T}(E)$. We have $b \in E_0$ and by (5) also $a \in E_0$, whence

$$a + \lambda b \in \hat{E}_0 \quad \text{for} \quad \lambda \in \mathbb{R} \tag{7}$$

since $E_0$ is a linear space. Relations (5)-(7) imply (2), which means according to (3) that $a \in \operatorname{core} A$ and (due to the arbitrariness of $a \in A$) $\operatorname{core} A = A$. Thus $A \in \mathcal{T}(E_0)$ and consequently

$$\mathcal{T}(E)|E_0 \subset \mathcal{T}(E_0). \tag{8}$$

Now take an arbitary $A \in \mathcal{T}(E_0)$. There exists a linear subspace $E_1$ of $E$ such that

$$E = E_0 \times E_1. \tag{9}$$

Put $\hat{A} = A \times E_1$. Then we have (5) and it is easy to check that $\hat{A} \in \mathcal{T}(E)$. Consequently $A \in \mathcal{T}(E)|E_0$, which implies that $\mathcal{T}(E_0) \subset \mathcal{T}(E)|E_0$. Together with (8) this yields (ii).

**Remark**. Claiming the existence of a linear space $E$, fulfilling (9) we have appealed, in fact, to the axiom of choice. The axiom of choice is assumed to be valid throughout this paper and will not be mentioned in the sequel.

**Lemma 3.** Let $E$, $X$ and $Y$ be linear spaces endowed with topologies $\mathcal{T}_E$, $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively, and let $E_0 \subset E$ be a linear subspace of $E$. Then:

i) If the topology $\mathcal{T}_E$ is linear [semilineaar], then also the topology $\mathcal{T}_E|E_0$ is linear [semilinear].

ii) If the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ are linear [semilinear], then also the topology $\mathcal{T}_X \times \mathcal{T}_Y$ is linear [semilinear].

*Proof.* Again we prove only (ii) for semilinear topologies. Put $\mathcal{T} := \mathcal{T}_X \times \mathcal{T}_Y$, $\mathcal{T}_0 := \mathcal{T}(\mathbb{R})$ (cf. Lemma 1) and

$$Z = X \times Y, \tag{10}$$

and let $\varphi : \mathbb{R} \times Z \times Z \longrightarrow Z$ be given by (1). By $\tilde{\varphi} : Z \longrightarrow Z$ and $\tilde{\tilde{\varphi}} : \mathbb{R} \longrightarrow Z$ we denote function (1) with the variables $\lambda$, $b$ resp. $a, b$ fixed.

Let $A \subset Z$ be an arbitrary set belonging to $\mathcal{T}$. In order to prove that $\mathcal{T}$ is semilinear it is enough to show that for $\lambda \in R$ and $b \in Z$ arbitrarily fixed we have

$$\tilde{\varphi}^{-1}(A) \in \mathcal{T} \tag{11}$$

and for $a, b \in Z$ arbitrarily fixed we have

$$\tilde{\tilde{\varphi}}^{-1}(A) \in \mathcal{T}_0 \tag{12}$$

(The continuity of $\varphi$ with respect to $b$ is a consequence of the continuity of $\varphi$ with respect to $a$ and of the commutativity of addition). Moreover, we may restrict ourselves to sets $A$ from the neighbourhood base of $\mathcal{T}$, that is, to sets $A$ of the form $A = G \times H$, where $G \in \mathcal{T}_X$, $H \in \mathcal{T}_Y$.

According to (10) the points $a, b \in Z$ can be written as $a = (a_x, a_y)$ and $b = (b_x, b_y)$, where $a_x, b_x \in X$, $a_y, b_y \in Y$. We define functions $\varphi_x : \mathbb{R} \times X \times X \longrightarrow X$ and $\varphi_y : \mathbb{R} \times Y \times Y \longrightarrow Y$ by

$$\varphi_x(\lambda, a_x, b_x) = \lambda a_x + b_x, \quad \varphi_y(\lambda, a_y, b_y) = \lambda a_y + b_y, \tag{13}$$

and tildes and double tildes applied to $\varphi_x$ and $\varphi_y$ have analogous meaning as in the case of $\varphi$. By (1) and (13)

$$\begin{aligned}\varphi(\lambda, a, b) &= \lambda a + b = \lambda(a_x, a_y) + (b_x, b_y) = (\lambda a_x + b_x, \lambda a_y + b_y)\\ &= (\varphi_x(\lambda, a_x, b_x), \varphi_y(\lambda, a_y, b_y))\end{aligned}$$

so that with $\lambda \in \mathbb{R}$ and $b = (b_x, b_y) \in Z$ fixed we have for $A = G \times H$

$$\tilde{\varphi}^{-1}(A) = \tilde{\varphi}_x^{-1}(G) \times \tilde{\varphi}_y^{-1}(H), \tag{14}$$

while with $a = (a_x, a_y)$, $b = (b_x, b_y) \in Z$ fixed we have for $A = G \times H$

$$\tilde{\tilde{\varphi}}^{-1}(A) = \tilde{\tilde{\varphi}}_x^{-1}(G) \times \tilde{\tilde{\varphi}}_y^{-1}(H), \tag{15}$$

Now, tilded and double tilded functions (13) are continuous, since the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ are semilinear. Thus (11) and (12) are a consequence of (14) and (15).

Now we prove our main result.

**Theorem 1.** Let $X$ and $Y$ be linear spaces such that

$$\max(\dim X, \dim Y) \geq 2, \tag{16}$$

and let $Z$ be defined by (10). The topology $\mathcal{T} := \mathcal{T}(X)\times\mathcal{T}(Y)$ in $Z$ is semilinear, but is not linear and is different from the core topology $\mathcal{T}(Z)$.

*Proof.* $\mathcal{T}$ is semilinear by virtue of Lemmas 1 and 3 (ii). If $\mathcal{T}$ were linear, then, by (10) and Lemmas 2 (i) and 3 (i), also $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ would be linear, which however, in view of Lemma 1 is incompatible with (16).

Now suppose that $\mathcal{T} = \mathcal{T}(Z)$, that is

$$\mathcal{T}(X \times Y) = \mathcal{T}(X) \times \mathcal{T}(Y).$$

Fix arbitrary $u \in X \setminus \{0\}$ and $v \in Y \setminus \{0\}$ and write

$$L_u := \{x \in X \mid x = \lambda u, \lambda \in \mathbb{R}\}, \quad L_v := \{y \in Y \mid y = \lambda v, \lambda \in \mathbb{R}\}.$$

Thus $L_u$ and $L_v$ are one dimensional linear subspaces of $X$ and $Y$, respectively. By Lemma 2 (ii)

$$\mathcal{T}(X)|L_u = \mathcal{T}(L_u), \quad \mathcal{T}(Y)|L_v = \mathcal{T}(L_v), \quad \mathcal{T}(X \times Y)|(L_u \times L_v) = \mathcal{T}(L_u \times L_v), \tag{18}$$

whence by Lemma 2 (iii)

$$(\mathcal{T}(X) \times \mathcal{T}(Y))|(L_u \times L_v) = \mathcal{T}(L_u) \times \mathcal{T}(L_v). \tag{19}$$

Relations (17), (18) and (19) imply that

$$\mathcal{T}(L_u \times L_v) = \mathcal{T}(L_u) \times \mathcal{T}(L_v). \tag{20}$$

Since $\dim L_u = \dim L_v = 1$, the topologies $\mathcal{T}(L_u)$ and $\mathcal{T}(L_v)$ are linear and hence also $\mathcal{T}(L_u) \times \mathcal{T}(L_v)$ is linear (Lemmas 1 and 3 (ii)). On the other hand, we have $\dim(L_u \times L_v) = 2$ and thus, according to Lemma 1, the topology $\mathcal{T}(L_u \times L_v)$ is not linear. This contradiction shows that (20), and hence also (17), is impossible and completes the proof of the theorem.

**Corollary 1.** Let $X$, $Y$ be linear spaces. For arbitrary topologies $\mathcal{T}_X$ in $X$ and $\mathcal{T}_Y$ in $Y$ we always have

$$\mathcal{T}(X \times Y) \neq \mathcal{T}_X \times \mathcal{T}_Y. \tag{21}$$

In other words, the core topology never is a product topology.

*Proof.* Supposing the contrary we have

$$\mathcal{T}(X \times Y) = \mathcal{T}_X \times \mathcal{T}_Y, \tag{22}$$

whence we obtain by virtue of Lemma 2 (i) and (ii)

$$\mathcal{T}_X = (\mathcal{T}_X \times \mathcal{T}_Y)|X = \mathcal{T}(X \times Y)|X = \mathcal{T}(X) \tag{23}$$

and similarly

$$\mathcal{T}_Y = (\mathcal{T}_X \times \mathcal{T}_Y)|Y = \mathcal{T}(X \times Y)|Y = \mathcal{T}(Y). \tag{24}$$

Thus (22) becomes identical with (17).

Now, if $X$ and $Y$ fulfil (16), then (21) results from Theorem 1 in view of (23) and (24), and if $\dim X = \dim Y = 1$, then we argue as in the proof of Theorem 1 to show that relations (20) is impossible.

**Corollary 2.** Every linear space $Z$ such that

$$\dim Z \geq 3 \tag{25}$$

admits a semilinear topology which is neither linear nor identical with the core topology $\mathcal{T}(Z)$.

*Proof.* There exist linear spaces $X$ and $Y$ fulfilling (10), and condition (25) implies (16). Thus it follows from Theorem 1 that the topology $\mathcal{T}(X) \times \mathcal{T}(Y)$ in $Z$ $\varphi$ has all the required properties.

If $\dim Z = 2$, then $Z$ is isomorphic (may be identified) with $\mathbb{R}^2$. Endowed with the natural topology of the plane, $\mathbb{R}^2$ is a linear topological Baire and Hausdorff space satisfying the second axiom of countability. Thus we obtain from Corollary 2 and from the result of Z. Kominek [1] mentioned earlier in this paper the following

**Corollary 3.** Under the assumption of the continuum hypothesis every linear space $Z$ of dimension at least 2 admits a semilinear topology which is neither linear nor identical with the core topology $\mathcal{T}(Z)$.

In spaces $Z$ of high dimension we can presumably obtain several semilinear topologies applying Theorem 1 with different representations (10). We note also that having at our disposal some semilinear topologies we can generate further ones with the aid of the following generalization of Theorem 1.

**Theorem 2.** Let $X$ and $Y$ be linear spaces and let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be semilinear topologies in $X$ and in $Y$, respectively, at least one of which is not linear. Define $Z$ by (10). Then $\mathcal{T} := \mathcal{T}_X \times \mathcal{T}_Y$ is a semilinear topology in $Z$, which is neither linear nor identical with the core topology $\mathcal{T}(Z)$.

*Proof.* The topology $\mathcal{T}$ is semilinear by virtue of Lemma 3 (ii). If the topology $\mathcal{T}$ were linear, then, according to Lemmas 2 (i) and 3 (i), also both topologies $\mathcal{T}_X = \mathcal{T}|X$ and $\mathcal{T}_Y = \mathcal{T}|Y$ would be linear, which is not the case. Relation (21) results from Corollary 1.

Observe that if $\mathcal{T}_X$ and $\hat{\mathcal{T}}_X$ are (semilinear) topologies in a (linear) space $X$, $\mathcal{T}_Y$ and $\hat{\mathcal{T}}_Y$ are (semilinear) topologies in a (linear) space $Y$, and $\mathcal{T}_X \neq \hat{\mathcal{T}}_X$ and or $\mathcal{T}_Y \neq \hat{\mathcal{T}}_Y$, then $\mathcal{T}_X \times \mathcal{T}_Y \neq \hat{\mathcal{T}}_X \times \hat{\mathcal{T}}_Y$ are different (semilinear) topologies in the (linear) space $Z = X \times Y$. This is consequence of Lemma 2 (i).

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