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AN AXIOMATIZATION OF FUZZY CLASSES

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ABSTRACT

An axiomatization of fuzzy classes more general than usual Fuzzy Sets is proposed. Connections and interpretations with other axiomatizations of Set Theory and Fuzzy Set Theory are investigated.

Keywords: Axiomatic Theory, Classical Set Theory, Interpretation.

Introduction.

Some axiomatizations of fuzzy sets have already been proposed: [2] (see also the continuation [3]), [8], [12]; unfortunately the theory [8] is not sufficiently developed to be judged, while the other two axiomatize Fuzzy Sets of Zadeh [13] (see also [1], [6]), with some differences between them and paralleling the axiomatization of Zermelo-Fraenkel.

As noted in [5] there is some lack of critical and foundational discussions on Fuzzy Set theory; [5] itself and issue n. 1 (1988) of the review "Fuzzy Set and System" can be considered a starting point for it and [9] a more detailled discussion of criticisms and motivations compelling us to look for a new axiomatization of more general classes than Fuzzy Sets translating the actual vagueness.

To summarize, vague classes of everyday language are structured as Fuzzy Sets with membership degrees and valuation class, but also seem to satisfy the following properties:

1) different classes can have different valuation classes;
2) a valuation class is as fuzzy as the other classes (in particular it has membership degrees and a valuation class);

3) there is only one definition of union, intersection and complementation;

4) membership degrees in a valuation class are "positive".

(that is there is no minimum degree 0 so that if $\mu_Y(x) = 0$ then we say that x does not belong to Y, then an intuitive interpretation of the objects are functions with range the interval open to the left (0,1]).

The axiomatizations proposed in [2], [12], do not satisfy these conditions since the valuation class is fixed (even if it is not explicitly written), and it is not so fuzzy (in some sense) as the other classes since [2] and [12] put two crisp comparison relations on degrees. Neither 3) is satisfied since they set only one of the possible definitions of union and intersection (even if the most natural one). The two axiomatizations are also highly non- constructive: it is true that with the respective replacement axioms we can have sets in which membership degree are known and/or fixed, but, if we have two classes X and Y such that $Y \subseteq X$, in general we don't know what the membership degrees of Y can be even if we know those of X.

Together with these properties we could also assume that every valuation object is partially ordered. We choose instead to axiomatize a situation of maximal uncertainty where we can say only that x belongs to y with one and only one membership degree that is not further on specified: then considerations of ordering on the valuation classes can be omitted, even if, as we will see, an appropiate axiom of "ordering of the valuation classes" can be added to the theory we propose.

Assuming an axiom of construction, union, intersection and complementation would be uniquely determined by the propositional connectives, but, for example in $Y \cap Z$, we can say only that x belongs to $Y \cap Z$ (with some membership degree) if and only if it belongs to Y (with some degree) and to Z (with some (other) degree).

If we want different valuation classes it seems impossible to assume extensionality on every pair of objects, and the axiom of construction we assume has a particular form to take in consideration this fact.

A nucleus of classes satisfying axioms of classic set theory can be assumed.

Inside the theory we propose, Fuzzy Sets "à la Zadeh" are defined justifying, with the definition itself, the fact that different notions of union and intersection exist.

We will see tht our theory can be interpreted in KM, ZF and the theories of [2] and [12] can be interpreted in it.

I. The language and axioms.

1. **Metadefinition.**

a) The language we use is the (first order) language $\mathcal{L} = (\in, \equiv, \{,./.\})$, where $\equiv$ is the equality, $\in(.,.,.,.)$ is a quaternary predicate representing membership and $\{./.\}$ is the abstraction operator (see [7]).

b) Objects considered by the theory are "classes" and are indicated by a capital Latin letter.

c) We read $\in(X,Y,Z,W)$ as "X belongs to Y with (membership) degree Z respect to (the valuation object) W".

The following three axioms describe the structure of the classes.

2. **Axiom 1.** $(\forall Y)[((\exists X,Z,W) \in (X,Y,Z,W)) \longrightarrow (\forall V,V')$
$[(\exists N,N',T,T')(\in (T,Y,N,V) \wedge \in (T',Y,N',V')) \longrightarrow V \equiv V']).$

3. **Axiom 2.** $((\forall X,Y)[((\exists Z,W) \in (X,Y,Z,W)) \longrightarrow (\forall T,T')[(\exists W)(\in (X,Y,T,W) \wedge$
$\in (X,Y,T',W)) \longrightarrow T \equiv T']).$

4. **Axiom 3.** $((\forall Z,W)[((\exists X,Y) \in (X,Y,Z,W)) \longrightarrow ((\exists T,V)(\in (Z,W,T,V))].$

5. **Definition.**

a) A class X is a set, written Set(X), if $(\exists Y,Z,W) \in (X,Y,Z,W)$.

b) Given a class Y, a class W is its valuation class if and only if $(\exists X, Z) \in (X, Y, Z, W)$, and we write $\mathrm{Val}(Y) \equiv W$.

c) Given X, Y, Z, W such that $\in (X, Y, Z, W)$, we say that Z is the membership degree of X in Y (respect to W).

How we can give an intuitive interpretation of these axioms: with them we ask that:

1) if Y has at least one element then it has one and only one valuation class;

2) if X is an element of Y then X has one and only one membership degree in Y with respect to the valuation class of Y;

3) if Z is the membership degree of X in Y with respect to the valuation class W, then Z is an element of W (with a membership degree).

We note that, when in the following we have the empty set $\emptyset$, we will not be able to specify its valuation class by axiom 1: in fact we are not able to specify any membership degree in the empty set and therefore we can specify nothing about its valuation class.

Let us state now the axiom of construction of the theory. Its form is due to the fact that we cannot state an axiom of extensionality in the theory due to the structure of the classes. If we could define when $X \equiv Y$ then we should define at the same time when $\mathrm{Val}(X) \equiv \mathrm{Val}\ (Y)$, $\mathrm{Val}(\mathrm{Val}(X)) \equiv \mathrm{Val}(\mathrm{Val}(Y))$ and so on. We assume extensionality only on definable objects and in this way this way set constructions can be carried on (see after theorem 11 and the following observation).

Convention. A small letter indicates a set.

6. **Proposition.** If $\in (X, Y, Z, W)$ then Z is a set.

Proof. By axiom 3.

7. **Definition.** If $\varphi(X)$ is a formula of our language then the class that we find with the application of the abstraction operator to $\varphi(X)$ is indicated by $\{X/\mathrm{Set}(X) \wedge \varphi(X)\}$, or briefly, using small letters for sets $\{x/\varphi(x)\}$.

8. **Axiom 4.** Construction. Fixed a formula $\varphi(X)$, the following formula is an axiom:

$$(\forall y)[((\exists z, W) \in (y, \{y/\varphi(x)\}, z, W)) \longleftrightarrow \varphi(y)].$$

To simplify the notation we can introduce the following

Convention. From now on we will omit the valuation class since this is uniquely determined by axiom 1, so we write $\in (X, Y, Z)$ instead of $\in (X, Y, Z, \mathrm{Val}(Y))$, and so on.

With this convention our membership relation appears like those of [2] and [12]; but it remains essentially a quaternary relation.

We can now begin to build some classes of the theory, in particular we can define set-theoretical operations in the following points d-l as usual in classical set theory.

In the sequel we will also need the notion defined in m).

9. **Definition.**

a) $\emptyset$ is for $\{x/x \not\equiv x\}$.

b) If $b_1, \cdots, b_n$ are sets then $\{b_1, \cdots, b_n\}$ is (an abbreviation) for $\{x/x \equiv b_1 \vee \cdots \vee x \equiv b_n\}$.

c) V is for $\{x/x \equiv x\}$.

Given two classes X and Y, the:

d) union of X, $U(X)$, is for $\{z/(\exists r, r', s)(\in (z, s, r) \wedge \in (s, X, r'))\}$;

e) union of X and Y, $X \cup Y$, is for $\{z/\,(\exists t) \in (z, X, t) \vee (\exists r) \in (z, Y, r)\}$;

f) intersection of X and Y, $X \cap Y$, is for $\{z/(\exists t) \in (z, X, t) \wedge (\exists r) \in (z, Y, r)\}$;

g) difference of X and Y, $X - Y$, is for $\{z/(\exists t) \in (z, X, t) \wedge \neg((\exists r) \in (z, Y, r))\}$;

h) complement of X, $-X$, is for $\{z/\neg((\exists t) \in (z, X, t))\}$;

i) power of X, $\mathcal{P}(X)$, is for $\{z/z \subseteq X\}$;

l) Cartesian pair of X and Y, $< X, Y >$, is for $\{\{X\}, \{X, Y\}\}$.

m) We say that X is extensionally equal to Y, or X is equiextensional to Y, $X = Y$, if and only if $(\forall z)\,[(\exists t) \in (z, X, t) \longleftrightarrow (\exists r) \in (z, Y, r)]$.

X and Y are equiextensional if every x belonging to X (in some way) belongs to Y (in some possibly different way): if we interpret the objects of the theory as functions and we take $F : X \longrightarrow (0,1]$ and $G : Y \longrightarrow (0,1]$, then F and G are equiextensional if and only if $X = Y$.

To carry on set-construction we need some form of extensionality. We assume then the following axiom that assures that two different but equivalent constructions produce the same result.

10. **Axiom 5.** Fixed two formulas $\varphi(X)$ and $\psi(X)$ the following formula is an axiom:

$$(\forall y)(\varphi(y) \longleftrightarrow \psi(y)) \longrightarrow \{x/\varphi(x)\} \equiv \{x/\psi(x)\};$$

or equivalently $\{x/\varphi(x)\} = \{x/\psi(x)\} \longrightarrow \{x/\varphi(x)\} \equiv \{x/\psi(x)\}$.

We can now prove

11. **Theorem.**

a) Ru$\equiv \{x/\neg((\exists y) \in (x,x,y))\}$ is not a set.

b) $(\forall X)\, X = \{x/(\exists t \in (x,X,t)\}$.

c) $\{x,y\} \equiv \{y,x\}$.

d) If there are two formulas $\varphi(X)$ and $\psi(X)$ such that $Y \equiv \{x/\varphi(x)\}$ and $Z \equiv \{x/\psi(x)\}$, then $Y = Z \longrightarrow Y \equiv Z$.

Proof.

a) It is Russel's paradox.

b) By definition of $=$.

c) and d) are two examples of application of axiom 5.

Observation. It easy to see that the operations defined above satisfy the following properties as usual set operations, even if sometimes (as in the following 3), 6), 7), 9)) we can have only extensional equality and not equality if A is not defined by a formula, compare with the behaviour of Fuzzy Sets in [4];

1) commutativity $A \cup B \equiv B \cup A$, $A \cap B \equiv B \cap A$;
2) associativity $A \cup (B \cup C) \equiv (A \cup B) \cup C$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$;
3) idempotency $A \cup A = A$, $A \cap A = A$;
4) distributivity $A \cup (B \cap C) \equiv (A \cap B) \cup (A \cap C)$, $A \cap (B \cup C) \equiv (A \cup B) \cap (A \cup C)$;
5) $A \cap \emptyset \equiv \emptyset$, $A \cup V \equiv V$;
6) identity $A \cup \emptyset = A$, $A \cap V = A$;
7) absorption $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$;
8) De Morgan's law $-(A \cup B) \equiv -A \cap -B$, $-(A \cap B) \equiv -A \cup -B$;
9) involution $- - A = A$;
10) equivalence formula $(-A \cup B) \cap (A \cup -B) \equiv (-A \cap -B) \cup (A \cap B)$;
11) symmetrical difference formula $(-A \cap B) \cup (A \cap -B) \equiv (-A \cup -B) \cap (A \cup B)$;
12) excluded middle $A \cap -A \equiv \emptyset$, $A \cup -A \equiv V$.

Note that if there is a formula $\varphi(X)$ such that $A \equiv \{x/\varphi(x)\}$, then equality holds also in 3), 6), 7), 9). Note also that the law of excluded middle is satisfied as some authors hope, see [11].

In our theory the valuation class is essential to define well bounded (classic, clear or non-fuzzy) classes: these are defined as classes having a uniquely determined membership degree.

12. **Definition.**

a) A class X is well bounded (briefly $W - B$-class) if it is $\emptyset$, or its valuation class is a singleton: that is

$$WB(X) \text{ is for } X \equiv \emptyset \vee (\exists y)\,\mathrm{Val}(X) \equiv \{y\}.$$

b) If a class Y is well bounded, and $\mathrm{Val}(Y) \equiv \{x\}$, then instead of $\in (X, Y, x, \{x\})$ we write $X \in Y$.

Point b) signifies that in the case of well bounded class we can forget fuzziness.

Convention. We write $W - B$-class instead of well bounded class.

In the sequel we will come back to consider $W - B$-classes.

13. **Definition.** X is contained in Y, $X \subseteq Y$, if and only if $(\forall x)((\exists y) \in (x, X, y) \longrightarrow (\exists y') \in (x, Y, y'))$.

For a justification of the preceeding definition see [9]; note that $\subseteq$ comes directly from the connective $\longrightarrow$, that is $(A \subseteq B \longleftrightarrow (\forall x)((\exists t) \in (x, A, t) \longrightarrow (\exists t') \in (x, B, t')))$. For a comparison take two Fuzzy Sets $F : X \longrightarrow [0,1]$, and $G : Y \longrightarrow [0,1]$, then $F \subseteq G$ if and only if $X \subseteq Y$.

From now on our axiomatization follows closely the axiomatization of Kelley- Morse theory.

14. **Axiom 6.** Set($\emptyset$).

15. **Axiom 7.** Power.

$$(\forall x)(\exists y)(\forall z)[((\exists t) \in (z, y, t)) \longleftrightarrow z \subseteq x].$$

16. **Axiom 8.** Pair.

$$(\forall x, y)(\exists z)(\forall t)[((\exists r) \in (t, z, r)) \longleftrightarrow (t \equiv x \vee t \equiv y)].$$

17. **Axiom 9.** Union.

$$(\forall x)(\exists y)(\forall z)[((\exists t) \in (z, y, t)) \longleftrightarrow (\exists r, r', s)(\in (z, s, r) \wedge \in (s, x, r'))].$$

18. **Axiom 10.** Infinite.

$$(\exists x)[(\exists z) \in (\emptyset, x, z) \wedge (\forall y)((\exists t)(\in (y, x, t) \longrightarrow (\exists t') \in (y, \cup\{y\}, x, t'))].$$

We can also define as usual:

19. **Definition.**

a) A class X is a relation, Rel(X), if and only if its elements are cartesian pairs.

b) A class F is a function, Fnc(F), if and only if

$\mathrm{Rel}(F) \wedge (\forall x, y, y')((\exists z, z')(\in (<x, y>, F, z) \wedge \in (<x, y'>, F, z')) \longrightarrow y \equiv y')$.

c) If F is a function then the F-image of X, $F[X]$, is the class

$\{x/(\exists y, v, t)(\in (y, X, t) \wedge \in (<y, z>, F, v))\}$.

By definition we have that if X is extensionally equal to a relation then X is a relation.

The situation here is similar to [2]: a function is fuzzy in membership degrees, while the elements of a function are cartesian pairs, and two cartesian pairs of F having the same first element have also the same second element. Note that [12] does not define functions.

20. **Axiom 11.** Replacement.

$$(\forall x)(\forall F)(\mathrm{Fnc}(F) \longrightarrow \mathrm{Set}(F[x]))).$$

If we want $W-B$-classes to form a nucleus of non-fuzzy objects satisfying the axioms of ZF, we must assume the following three axioms.

21. **Axiom 12.** Extensionality on $W-B$-classes.

$$(\forall X, Y)[(WB(X) \vee WB(Y)) \longrightarrow (X \equiv Y \longleftrightarrow X = Y)].$$

We do not want, in general, two classes that are extensionally equal to be equal: for example two Fuzzy Sets of Zadeh can be different even if their universes are equal.

22. **Axiom 13.**

13.1) $WB(X) \longrightarrow WB(\mathcal{P}(X))$;

13.2) $(WB(X) \wedge WB(Y)) \longrightarrow WB(\{X, Y\})$;

13.3) $WB(X) \longrightarrow WB(\cup(X))$;

13.4) $(\exists x)(WB(x) \wedge \emptyset \in x \wedge (\forall y)(y \in x \longrightarrow y \cup \{y\} \in x))$;

13.5) $(WB(F) \wedge WB(X)) \longrightarrow (Fnc(F) \longrightarrow WB(F[X])))$.

That is, set-theoretical operations on $W-B$-classes produce $W-B$-classes.

23. **Axiom 14.** Regularity on $W-B$-classes.

$$(\forall X((WB(X) \wedge X \neq \emptyset) \longrightarrow (\exists x)(x \in X \wedge x \cap X \equiv \emptyset)).$$

We have finished the list of axioms of the theory, we will see later an itneresting axiom that can be added to these.

Metadefinition. The theory with the axioms:

a) 1) – 14) above is called Fuz;

b) 1) – 11) above is called Fuz^{-}.

25 Metatheorem.

a) Fuzz^{-} is interpretable in KM;

b) Fuz is interpretable in KM and ZF can be interpreted in Fuz.

Proof. a) We obtain an interpretation of Fuz^{-} in KM translating:

$$X \equiv Y \,(\text{in Fuz}^{-}) \text{ in } X = Y \,(\text{in } KM), \qquad \text{and}$$

$$\in (X, Y, Z, W)\,(\text{in } Fuz^{-}) \text{ in } (Z = \emptyset \wedge W = \{\emptyset\} \wedge X \in Y)\,(\text{in } KM).$$

By definition axiom 1 and 2 are satisfied since third and fourth elements of $\in (.,.,.,.)$ are uniquely determined.

If φ is a formula of Fuz, $\varphi^{\neg}$ is the interpretation of the formula in KM. Then we note that KM $\models [(\text{Set}(X))^{\neg} \longleftrightarrow (\exists Y)\, X \in Y]$, then sets are interpreted in sets; by this we observe that $< \emptyset >$ (in KM) coincides with $\{X/X \equiv \emptyset\}$ of Fuz. Now axiom 3 is satisfied since, by definition, $\in (\emptyset, \{\emptyset\}, \emptyset, \{\emptyset\})$, and the other ones are trivially verified.

b) The first part is as in point a): note also that the classes of the interpretation are well bounded ones. ZF is interpretable in Fuz since the class of well bounded sets satisfies the axioms of ZF.

Interpreting in the same way Fuz in ZF + urelements (atoms) + inaccessible we have an interpretation where there exist classes that are not well bounded. If we obtain a model of Fuz in KM we can add also classes in such a way that there is no W-B class in the model.

II. One more axiom. Classic objects and an interpretation of naive Fuzzy Sets.

1. **Definition.**

a) A relation R partial orders A, $PO(R, A)$, if and only if

$(\forall x, y, z)\,[[(\exists w, t, v)\,(\in (x, A, w) \vee \in (y, A, t) \vee \in (z, A, v))] \longrightarrow$

$[(\exists t) \in (< x, x >, R, t) \vee [(\exists r, s)(\in (< x, y >, R, r) \vee \in (< y, x >, R, s)) \longrightarrow$

$x \equiv y] \vee [(\exists r, s)(\in (< x, y >, R, r) \vee \in (< y, z >, R, s)) \longrightarrow (\exists q) \in (< x, z >, R, q)]]]$.

Now if R is a relation:

b) Dom(R) is for $\{x / (\exists z, t) \in (< x, z >, R, t)\}$.

c) Rng(R) is for $\{x / (\exists z, t) \in (< z, x >, R, t)\}$.

We can define another inclusion similar to the one of Zadeh.

2. **Definition.** Given two classes X and Y, if Val(X) $\equiv$Val(Y) and $PO(R,\text{Val}(X))$, then $X \leq_R Y$ is for $(\forall x)[(\exists y) \in (x, X, y) \longrightarrow ((\exists z) \in (x, Y, z) \wedge (\exists t) \in (< y, z >, R, t))]$.

Note that the relation $\leq$ depends on R.

We can also state another axiom that can be added to Fuz.

3. **Axiom 15.**

$$(\forall X)[((\exists Y)\text{Val}(Y) \equiv X) \longrightarrow ((\exists R)PO(R, X))].$$

Note that for Fuz + Ax15 the metatheorem I.25 is always valid.

As noted in theorem I.25, we have inside Fuz an obvious model of ZF, the class of well bounded sets. We can define notions of naive Fuzzy Set theory inside Fuz. From now on we work in Fuz and classes will be well bounded ones.

Convention. If R is a well bounded relation then we write xRy instead of $< x, y > \in R$.

Obviously, a well bounded relation R partial orders a $W - B$-class A if and only if $(\forall x, y, z \in A)[xRx \wedge ((xRy \wedge yRx) \longrightarrow x \equiv y) \wedge ((xRy \wedge yRz) \longrightarrow xRz)]$.

4. **Definition.**

a) A partially ordered set is a set x such that there is a relation R and $PO(R, x)$.

b) Fixed two sets x and v, with w partially ordered set, a support of an universe of approximation of degree 1 on x is the class $\{y/(\exists a)y =< a, x, w > \wedge \text{Fnc}(a) \wedge \text{Dom}(a) = x \wedge \text{Rng}(a) \subseteq w\}$, written $\mathcal{U}(w, x, 1)$, its elements are called approximations.

c) If $< a, x, w >$ and $< b, x, w >$ are elements of the same $\mathcal{U}(w, x, 1)$ we say that $< a, x, w >$ is contained in $< b, x, w >$, $< a, x, w > \preceq < b, x, w >$, if and only if for every y, $a(y)Rb(y)$, where R is the partial order of w.

d) We have an universe of approximation if we have an $\mathcal{U}(w, x, 1)$ and two operations $\perp$ and $\top$ on $\mathcal{Y}(w, x, 1)$ such that for every pair of approximations $A \equiv < a, x, w >$ and $B \equiv < b, x, w >$, $A, B \preceq A \top B$, and $A \perp B \preceq A, B$.

e) Fixed two sets x and w, a support of an universe of approximation of degree 2 on x is the class $\{y/(\exists a)y =< a, x, \mathcal{U}(w, w, 1) > \wedge \text{Fnc}(a) \wedge \text{Dom}(a) \equiv x \wedge \text{Rng}(a) \subseteq \mathcal{U}(w, w, 1)\}$.

Note that $\mathcal{U}([0, 1], x, 2)$ are just type 2 Fuzzy Sets (see [4]). Naturally it is also possible to define universes of approximation of higher degrees.

Now we can catch the usual definitions of fuzzy Sets theory, see [4] and [13], that is:

Observation. Taking $\mathcal{U}([0, 1], x, 1)$, we obtain an universe of approximation setting

$$< a, x, [0, 1] > \perp < b, x, [0, 1] > = < c, x, [0, 1] >, \quad \text{and}$$

$$< a, x, [0, 1] > \top < b, x, [0, 1] > = < d, x, [0, 1] >,$$

where c and d are such that for every y, $c(y) = \inf\{a(y), b(y)\}$, and $d(y) = \sup\{a(y), b(y)\}$.

Similarly, we can obtain universes of approximation where $\perp$ and $\top$ are bold union and bold intersection, probabilistic sum and product.

When we have a real problem we choose then the best way of approximating it, i.e., we choose an universe of approximation.

We now compare Fuz with the other axiomatizations of Fuzzy Sets theory, see [2] and [12].

We know from [12] that

5. **Theorem.** The theory proposed in [12] is interpretable in ZF.

We know also from [10] that

6. **Theorem.**

a) ZF is interpretable in the theory proposed in [12].

b) The theory proposed in [2] is interpretable in ZF and ZF can be interpreted in it.

By these theorems and theorem I.25, we immediately obtain that:

7. **Theorem.** The theories proposed in [2] and [12], can be interpreted in Fuz.

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