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ON THE GENERATORS OF T-INDISTINGUISHABILITY OPERATOR*

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ABSTRACT

The structure of the generators' set of a T-indistinguishability operator is analyzed. A suitable characterization of such generators is given. T-indistinguishability operators generated by a single fuzzy set, in the sense of the representation theorem, are studied.

1. Introduction.

In Trillas [18] and Trillas & Valverde [19, 20], indistinguishability relations are introduced in order to give a unifying mathematical structure to the various concepts and models of "equality" arising from different fields that range from Psychology and Social Sciences to the Fuzzy Set Theory passing through Multivalued Logic or Probabilistic Metric Spaces theory.

Many results in Fuzzy Sets Theory have given a new light to the meaning and the applications of such relations. For instance, in this context Menger's probabilistic relations [12, 14] are revisited as the equivalence relations associated to a certain type of multivalued logics. Similarity relations, widely used in mathematical Taxonomy [4, 10] are Zadeh's Min-transitive relations [25], Ruspini's Likeness relations [15] are introduced in order to solve problems in Cluster Analysis and so on.

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All these aforementioned relations are examples of T-indistinguishability operators, i.e. reflexive, symmetric and T-transitive fuzzy relations, T being a continuous t-norm [1].

In Valverde [21, 22] a representation theorem for such relations is proved. This theorem allows the construction of T-transitive relations in a more efficient way than the transitive closure [22] and the graph theoretical methods, opening new paths to the applications of these relations.

In this paper, the functions that generate a given T-indistinguishability operator E are characterized and the structure of the family of fuzzy subsets that are generators of a given T-indistinguishability operator is analyzed.

For the sake of completeness, there is a preliminary section concerned with the definition and properties of T-indistinguishability operators. The standard notations and conventions related to fuzzy relations and t-norms will be used in this paper. For a detailed study of t-norms and t-conorms readers are referred to [17] and some specific properties related to their use in Fuzzy Set Theory can be found in [2, 3, 22].

2. Preliminaries.

In the sequel X stands for a non-empty set, T for a continuous t-norm and $\hat{T}$ for its quasi-inverse i.e. $\hat{T}(x|y) = \sup\{\alpha \in [0,1] | T(x,\alpha) \leq y\}$. A T-indistinguishability operator is defined in the following way:

Definition 2.1. A map E from $X \times X$ into $[0,1]$ is termed *T-indistinguishability operator* if the following properties hold for any x, y and z of X:

(2.1.1) $E(x,x) = 1$ (reflexitivity)

(2.1.2) $E(x,y) = E(y,x)$ (symmetry)

(2.1.3) $T(E(x,y), E(y,z)) \leq E(x,z)$ (T-transitivity).

Since Zadeh [24] a large number of papers dealing with different aspects connected to this type of relations have appeared. The weak point of all methods that use fuzzy transitive relations relays in its effective construction. The only existing methods until 1982 where the transitive closure method or the methods based on Graph Theory. Both techniques present important computacional difficulties [9, 22] and they require to start with a reflexive and symmetric relation. In Valverde [21] the following representation theorem is proved:

Theorem 2.1 (Representation theorem). A map E from $X \times X$ onto $[0,1]$ is a T-indistinguishability operator if, and only if there exists a family $\{h_j\}_{j\in J}$ of fuzzy subsets of X such that

$$E(x,y) = \inf_{j\in J} \hat{T}(\max(h_j(x), h_j(y))| \min(h_j(x), h_j(y))) \tag{2.1.4}$$

This theorem, which extends a previous result by S. Ovchinnikov [13], allows the construction of a T-transitive relation in one step and it only requires to start with a set of functions from X into $[0,1]$. Its constructive aspect and generality open new possibilities for a more profound study of its structure and applications [9].

If we turn our attention to the formula (2.1.4), we immediately observe two main constitutive elements i.e. the set $\{h_j\}_{j\in J}$ of functions that evaluate the elements of X and the t-norm T that modelizes the transitivity.

From this point of view, it is interesting to analyze the conditions that qualify a fuzzy set h to be an element of a family that generates a given T-indistinguishability operator E; if so, this function will be termed a **generator of E**. In the next section this conditions are studied and the structure of the generators' set is investigated.

On the other hand, if we define the relation

$$E_j(x,y) = \hat{T}(\max(h_j(x), h_j(y))| \min(h_j(x), h_j(y))) \tag{2.1.5}$$

then the Representation Theorem can reformulated as

$$E(x,y) = \inf_{j\in J} E_j(x,y) \tag{2.1.6}$$

that is: any T-indistinguishability operator is the infimum of a family of operators generated by a single function, this type of T-indistinguishability operators are studied in the last section.

3. The set of generators.

In this section, we characterize the generators of a given T-indistinguishability operator E and we study its structure. A generator is defined in the following natural way:

Definition 3.1. A function h from X into $[0,1]$ is termed a *generator* of a given T-indistinguishability operator E, if h is an element of any of the families $\{h_j\}_{j\in J}$ that generate E in the sense of the Representation Theorem.

H_E will denote the set of all generators of E. It follows immediately from the Representation Theorem that, given a T-indistinguishability operator E on X, the set $\{E(x,y)\}_{y\in X}$ of fuzzy subsets of X is a generating family of E and will be denoted by $\{h_y(x))\}_{y\in X}$.

It is also easy to prove that if E is generated by the family $\{h_j\}_{j\in J}$ and E_j, is the T-indistinguishability operator defined in (2.1.5) then $E_j(x,y) \geq E(x,y)$ for all x, y of X. Finally let us observe that if $h \in [0,1]^X$ is such that $E_h \geq E$ and $\{h_j\}_{j\in J}$ is a generating family of E then the set $\{h_j\}_{j\in J} \cup \{h\}$ also generates E.

The next definition will play an important role in order to give a more convenient characterization of the generators of a T-indistinguishability operator E.

Definition 3.2. If E be a T-indistinguishability operator then ϕ_E is the map from $[0,1]^X$ into $[0,1]^X$ defined by

$$\phi_E(h)(x) = \sup_{y\in X}\{T(E(x,y),h(y))\}, \text{ for any } x \in X$$

It is worth noting that if X is a finite set then E is represented by a matrix and ϕ_E may be understood as the max-T product of E by the column vector representing the fuzzy set h.

Theorem 3.1. If ϕ_E is the map associated to a T- indistinguishability operator E, then the following properties hold:

i) If $h_1 \geq h_2$ then $\phi_E(h_1) \geq \phi_E(h_2)$ i.e. ϕ_E is an increasing function with respect to the pointwise order in $[0,1]^X$ and $\phi_E \geq h$ for any $h \in [0,1]^X$.

ii) $\phi_E(h_1 \vee h_2) = \phi_E(h_1) \vee \phi_E(h_2)$, i.e. ϕ_E is un V-morphism in the lattice $\{[0,1]^X, \max, \min\}$

iii) $(\phi_E)^2(h) = \phi_E(h)$, i.e. ϕ_E is an idempotent map.

Proof. The parts i) and ii) follow immediately from the monotony of the t-norm T.

iii) If $h' = \phi_E(h)$ then,

$\phi_E(h')(x) = \sup_{y\in X} T(E(x,y), h'(y)) =$

$= \sup_{y\in X} T(E(x,y), \sup_{z\in X} T(E(y,z), h(z))) =$

$= \sup_{y\in X}(\sup_{z\in X} T(x,y), T(E(y,z), h(z))) =$

$= \sup_{y\in X}(\sup_{z\in X} T(T(E(x,y), (E(y,z), h(z))) \leq$

$\leq \sup_{y\in X}(\sup_{z\in X} T(E(x,z), h(z))) = \sup_{y\in X}(h'(x)) = h'(x)$

and as $\phi_E(h')(x) \geq h'(x)$ then $(\phi_E)^2(h)(x) = \phi_E(h)(x)$.

Theorem 3.2. (Generators characterization). A fuzzy subset $h \in [0,1]^X$ is a generator of a T-indistinguishability operator E if, and only if $\phi_E(h) = h$.

Proof. From Theorem 3.1.-i), $\phi_E(h)(x) \geq h(x)$ for any $h \in [0,1]^x$.

On the other hand, if h is a generator then, for any x, y of X

$$E_h(x,y) \geq E(x,y)$$

and, therefore,

$T(E(x,y), h(y)) \leq (T(E_h(x,y), h(y))) =$

$T(\hat{T}(\max(h(x), h(y))| \min(h(x), h(y))), h(y))) \leq h(y)$

for all y in X. Consequently,

$$\phi_E(h)(x) = \sup_{y \in X} T(E(x,y), h(y)) \leq h(x)$$

for all x in X, i.e. $\phi_E(h)(x) = h(x)$.

Reciprocally, if $\phi_E(h)(x) = h(x)$, for any x , y in X, both

$$T(E(x,y), h(y)) \leq h(x) \quad \text{and}$$

$$T(E(x,y), h(x)) \leq h(y) \quad \text{hold}$$

From those inequalities it follows that

$$\hat{T}(h(x)|h(y)) \geq E(x,y) \qquad \text{and}$$

$$\hat{T}(h(y)|h(x)) \geq E(x,y), \qquad \text{i.e.}$$

$$\hat{T}(\max(h(x), h(y))| \min(h(x), h(y))) \geq E(x,y) \quad \text{thus, } E_h \geq E$$

consequently, h is a generator of E.

From the previous theorems it follows:

Proposition 3.1. If E is a T-indistinguishability operator then:

i) $H_E = \phi_E([0,1]^X)$

i)) If $h_k \in [0,1]^x$ is a constant fuzzy set i.e. $h_k(x) = k$ for any $x \in X$, then $h_k \in H_E$.

iii) If $h_1, h_2 \in H_E$ then $\min(h_1, h_2) \in H_E$.

Proof. The parts i-ii are immediate, and part iii follows from the monotony of the t-norm and theorem 3.1.-i.

From this proposition, it immediately follows:

Corollary 3.1. $\{H_E, \max, \min\}$ is a sub-lattice of the lattice $\{[0,1]^X, \max, \min\}$ with supremum h_1 and infimum h_0 which are constant fuzzy sets with values 0 and 1 respectively.

Taking into account Definition 3.1, the following proposition can be easily shown.

Proposition 3.2. If h^α is a classical singleton, i.e. $h^\alpha(x) = 1$ if $x = \alpha$ and $h^\alpha(x) = 0$ otherwise, then $\phi_E(h^\alpha)(x) = E(x, \alpha)$.

If X is a finite set, the columns of the matrix associated to E are the images of the classical singletons and we have

Corollary 3.2. $\{\phi_E(2^X), \vee\}$ is a $\vee$-semilattice generated by $\{E(x,y)\}_{y \in X}$.

The image of a classical set is obtained as the supremum of elements of the set $\{E(x,y)\}_{y \in X}$.

The constant fuzzy sets play a basic role in order to generate H_E, as it is shown in the next propositions.

Proposition 3.3. If h and $h_k \in [0,1]^X$ where $h_k(x) = k$, then

$$\phi_E(T(h)(x), h_k) = T(\phi_E(h)(x), h_k)$$

From the last result, it is easy to prove:

Proposition 3.4. The family of fuzzy sets $\{k_y(x) | k \in [0,1],\ y \in X\}$, where $k_y(x) = T(E(x,y), k)$, generates H_E considered as a V-semilattice.

4. Unidimensional T-indistinguishabilities.

This section is devoted to the study and characterization of T- indistinguishability operators generated by a single function h.

From now on, in order to simplify the proofs, only T-indistinguishability operators E such that, $E(x,y) \neq 1$ if $x \neq y$, will be considered. Anyway, if $\simeq$ is the equivalence relation in X, $x \simeq y(E)$ if, and only if $E(x,y) = 1$, then the induced T-indistinguishability operator on quotient set $\tilde{X} = X/\simeq$ satifies the mentioned condition.

Lema 4.1. If E is generated by a function h, then h is injective.

Proof. It follows immediately from the Representation Theorem.

In the sequel, it will also be assumed that $h \in [0,1]^X$ has a maximum and a minimum in X. The points of X where these extrem values are reached will be denoted by x_M and x_m, that is:

$$h(x_M) = \max\{h(x)|x \in X\} \quad \text{and} \quad h(x_m) = \min\{h(x)|x \in X\}.$$

Under this hypothesis we have the following lemma

Lema 4.2. If E is generated by h, then

$$E(x_M, x_m) = \min\{E(x,y)|x,y \in X\}$$

Proof. It follows from the properties of $\hat{T}$ [22].

Proposition 4.1. If E is generated by a single function, then,

$$T(E(x_M,y),E(x_m,y)) = E(x_M,x_m) \tag{4.1.1}$$

Proof. The transitivity of E entails

$$T(E(x_M,y),E(x_m,y)) \leq E(x_M,x_m)$$

that is

$$T(\hat{T}(h(xM)|h(y)),\hat{T}(h(y)|h(x_m)) \leq E(x_M,x_m)$$

If $T(x,y) = \min(x,y)$

$$E(x_M, y) \geq E(x_M, x_m) \quad \text{and} \quad E(x_m, y)) \geq E(x_M, x_m), \qquad \text{thus,}$$

$$\min(E(x_m, y)), E(x_M, y)) \geq E(x_M, x_m)$$

from where (4.1.1) follows at once.

If T is an archimedean and t its additive generator, then

$$\hat{T}(x|y) = t^{[-1]}(t(y) - t(x))$$

where t is a strictly decreasing function from $[0,1]$ into $[0,+\infty]$.

Under such hypothesis

$$T(E(x_M, y), E(x_m, y) = t^{[-1]}(t(t^{[-1]}(th(y) - th(x_M))) + t(t^{[-1]}(th(x_m) - th(y)))$$

If we assume that $y \neq x_M, x_m$ and $h(x_m) \neq 0$(*)

$$th(y) - th(x_M) < t(0) \quad \text{and} \quad th(x_m) - th(y) < 0,$$

then, for this values $t \circ t^{[-1]} = t \circ t^{-1} = j$, and

$$T(E(x_M, y), E(x_m, y) = t^{[-1]}(th(x_m) - th(x_M)) = E(x_M, x_m).$$

If E is generated by the function h, and $h_M(y) = E(x_M, y)$, $h_m(y) = E(x_m, y)$ with $E(x_M, x_m) > 0$, then we have the following

Proposition 4.2. If T is an archimedean t-norm then, E is generated by h_M or h_m both, h_M and h_m are generators of E.

(*)If T is a strict t-norm and $h(x_m) = 0$ then $th(x_m) = +\infty$ and $E(x_M, x_m) = 0$, $E(x_m, y) = 0$ for all $y \in X$, $y \in x_m$. In this case we consider the T-indistinguishability operator E' defined in $X' \times X'$ where $X' = X - \{x_m\}$.

Proof. It is an immediate consequence of the Archemedeanity of the t- norm and the Representation Theorem.

If T is the t-norm Min we have the following proposition

Proposition 4.3. If E is a similarity i.e. a T-indistinguishability operator with $T(x,y) = \min(x,y)$, then $h_M(x) = E(x_M, x)$ generates E.

Proof. If E_M is the T-indistinguishability operator generated by h_M, we have

$$E_M(x,y) = \overset{\frown}{\min}(E(x_M,x), E(x_M,y))$$

but, h_M is injective and therefore,

$$E(x_M,x) = \overset{\frown}{\min}(h(x_M),h(x)) = \min(h(x_M),h(x)) = h(x) \qquad x \neq x_M$$

$$E(x_M,y) = \overset{\frown}{\min}(h(x_M),h(y)) = \min(h(x_M),h(y)) = h(y) \qquad y \neq x_M$$

so,

$$E_M(x,y) = \overset{\frown}{\min}(h(x),h(y)) = E(x,y).$$

Proposition 4.4. If E is a unidimensional T-indistinguishability operator generated by $h \in [0,1]^X$, T is Archimedean and $\min\{E(x,y)|x,y \in X\} \neq 0$, then h_M and h_m are the only generating columns.

Proof. Let $h_{x'}(y) = E(x',y)$ be any other column of E, and $E_{x'}$ the T-indistinguishability operator generated by $h_{x'}$ then $h(x_m) < h(x') < h(x_M)$, and

$E_{x'}(x_m,x_M) = t^{[-1]}|tE(x',x_M) - tE(x',x_m)|$ but,

$E_{x'}(x',x_M) = t^{[-1]}th(x') - th(x_M)$ and

$E_{x'}(x_m,x') = t^{[-1]}th(x_m) - th(x'))$ therefore

$E_{x'}(x_m,x_M) = t^{[-1]}|t \circ t^{[-1]}(th(x') - th(x_m)) - t \circ t^{[-1]}(th(x_M) - th(x'))|$

$= t^{[-1]}(2th(x') - th(x_M) - th(x_m)) \neq E_x(x_m.x_M)$

consequently $h_{x'}$ is not a generator of E.

The following theorems give a characterization of the unidimensional T- indistinguishability operators that completes the results of S. V. Ovchinnikov [13] for probabilistic relations.

Let E be a T-indistinguishability opeator with T archimedean and t its additive generator such that $E(x, y) > 0$ for any $x, y \in X$.

Theorem 4.1. (Characterization theorem). A T-indistinguishability operator E is generated by a single function h if, and only if there exists a total order in X ($\leq_*$) with first element a and last element b, and such that for any x, y, z in X with $a \leq_* x, \leq_* y \leq_* z <_* b$

$$T(E(x,y), E(y,z)) = E(x,z) > 0 \tag{4.1.1}$$

Proof. If E is generated by h (injective), we define in X the order induced in it by the order of the reals in the unit interval

$$x \leq_* y \quad \text{if, and only if,} \quad h(x) \leq h(y)$$

So, for any x, y, z in X such that $a \leq_* x \leq_* y \leq_* z <_* b$

$$E(x,y) = \hat{T}(h(y)|h(x)) = t^{[-1]}(th(x) - th(y))$$

$$E(y,z) = \hat{T}(h(z)|h(y)) = t^{[-1]}(th(y) - th(z))$$

and taking in account that $E(x, y) > 0$ and $E(y, z) > 0$, then

$$t(E(x,y)) = th(x) - th(y), \quad t(E(y,z)) = th(y) - th(z)$$

therefore,

$$t(E(x,y) + t(E(y,z)) = th(x) - th(z) > 0$$

$$t^{[-1]}(tE(x,y) + tE(y,z)) = t^{[-1]}(th(x) - th(z)) = \hat{T}(h(z)|h(x)) = E(x,z) > 0$$

consequently

$$T(E(x,y),E(y,z)) = E(x,z)$$

Reciprocally, if there exists a total order in X $(\leq_*)$ and for any x, y, z in X such that $a \leq_* x \leq_* y \leq_* z <_* b$ condition (4.1.1.) holds, let $h(u) = E(a,u)$ with $u \in X$ and $a = \text{Imf } X$. For any v, w in X we have:

If $a \leq_* v \leq_* w$, then

$$T(E(a,v), E(v,w) = E(a,w) > 0 \quad \text{and} \quad E(v,w) = \hat{T}(E(a,v)E(a,w)),$$

and if $a \leq_* w \leq_* w$

we have

$$T(E(a,w), E(w,v)) = E(a,v) > 0 \qquad \text{thus,}$$

$$E(v,w) = \hat{T}(E(a,w)|E(a,v)).$$

Consequently

$$E(v,w) = \hat{T}(\max(h(v), h(w))|\min(h(v), h(w)))$$

and therefore E is generated by h.

Theorem 4.2. A similarity E is unidimensional if, and only if there exists $k \in X$ such that $h_k(x) = E(k,x)$ is injective.

Proof. The condition is obviously necessary. To prove the sufficient part, let $h_k(x) = E(h,x)$ be an injective function, and E_k the T- indistinguishability operator generated by h_k,

$$E_k(x,y) = \widehat{\min}(h_k(x), h_k(y)) \quad \text{for any } x, y \text{ in } X.$$

and, if $x \neq y$, the transitive property of E entails

$$E_k(x,y) = \min(E(k,x), E(k,y)) \leq E(x,y).$$

On the other hand, if h_k is a generator of E then

$$E_k(x,y) \geq E(x,y) \quad \text{thus } E_k(x,y) = E(x,y).$$

5. Summary.

The foregoing analysis is concerned with the study and characterization of the generators of T-indistinguishability operators in the sense of the Representation Theorem. The results contained in this paper open the path in order to summarize in a minimal set of generators the information contained in such operators.

On the other hand, the characterization of unidimensional relations, shows that these operators define a "betweeness" in the underlying set X, structured as a chain, giving a "geometrical" interpretation of the unidimensionality.

Finally, the existing duality betweee T-indistinguishability operators and a type of generalized metrics [22], leads to the interesting problem of the study of the topological structures induced by these metrics. The author intends to investigate this topic in a future publication.

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