Stochastica XII-1 (1988), 5-17

CONTRACTIONS ON PROBABILISTIC METRIC SPACES: EXAMPLES AND COUNTEREXAMPLES

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ABSTRACT

The notion of a contraction mapping for a probabilistic metric space recently introduced by T.L. Hicks is compared with the notion previously introduced by V.L. Sehgal and A.T. Bharucha-Reid. By means of appropriate examples, it is shown that these two notions are independent. It is further shown that every Hicks'contraction on a PM space $(S, \mathcal{F}, \tau_W)$ *is an ordinary metric contraction with respect to a naturally defined metric on that space; and it is again pointed out that, in Menger spaces under Min and similar t-norms, the contractions of Sehgal and Bharucha-Reid are also ordinary contractions on related metric spaces.*

1. Introduction.

In 1972, V.H. Sehgal and A.T. Bharucha-Reid [5] initiated the study of contraction mappings on probabilistic metric (briefly, PM) spaces. Their notion of a contraction, which we shall refer to as a B- contraction, is defined as follows:

Definition 1.1. A mapping f of a probabilistic semimetric (briefly, PSM) space $(S, \mathcal{F})$ into itself is a B-contraction if there is a γ in (0,1) such that for all points p, q in S and all $x > 0$,

$$F_{fp,fq}(\gamma x) \geq F_{pq}(x). \tag{1.1}$$

*Research supported in part by ONR Contract N-00014-87-K-0379.

In their paper Sehgal and Bharucha-Reid showed that any B-contraction on a complete Menger space $(S, \mathcal{F}, \tau_M)$ has a unique fixed point. Subsequently, the second author of the present paper showed that this result is the exception rather than the rule: specifically, for any Archimedean t- norm T, there exists a complete Menger space $(S, \mathcal{F}, \tau_T)$ and a B-contraction f on $(S, \mathcal{F})$ which has no fixed point [7].

Recently, T.L. Hicks [1] considered another notion of contraction mapping, which we shall refer to as an H-contraction, and which is obtained by replacing (1.1) by

$$F_{fp,fq}(\gamma x) > 1 - \gamma x \text{ whenever } F_{pq}(x) > 1 - x. \tag{1.2}$$

Hicks showed that any H-contraction on a complete Menger space $(S, \mathcal{F}, \tau_M)$ has a unique fixed point. He conjectured that H-contractions and B- contractions are distinct notions and asked whether the t-norm M in his fixed- point theorem could be replaced by a weaker one. This last question was answered by V. Radu who showed that any H-contraction on a complete Menger space $(S, \mathcal{F}, \tau_T)$ for which $\sup_{a<1} T(a,a) = 1$ has a unique fixed point [3].

The principal aim of this paper is to compare the notions of B-contraction and H-contraction with each other and with ordinary metric contractions. In particular, we construct examples which show that Hicks'conjecture is correct, i.e., that in general the two notions are independent. We also give two conditions which, together but not individually, are sufficient to guarantee that a B-contraction is an H-contraction. Then, by considering B-contractions and H-contractions on α-simple spaces and pseudometrically generated PM spaces, we show that both notions have their deficiencies. We conclude with several comments. In the course of our studies, we also identify a metric first defined for Menger spaces under Min by Hicks [1] and subsequently for Menger spaces under t-norms stronger than W, by Radu [3]. Throughout, we assume that the reader is familiar with the basic concepts and terminology of the theory of PM spaces as given, e.g., in [4].

2. H-contractions and metric contractions.

In this section we show that, subject to a rather mild restriction, every H-contraction on a PM space is an ordinary contraction with respect to a naturally defined metric on that space. Most of the results, in a somewhat different and less general formulation, are due to Radu [3]. To begin, we need the following result, which is Lemma 4.3.3 of [4].

Lemma 2.1. For any distribution function F in Δ^+ and any $t > 0$,

$$F(t) > 1 - t \text{ if and only if } d_L(F, \epsilon_0) < t. \tag{2.1}$$

The function d_L in (2.1) is the modified Lévy metric, but since

$$d_L(F, \epsilon_0) = \inf\{h|F(h+) > 1 - h\} \tag{2.2}$$

(see (4.3.3) of [4]), it is clear that (2.1) remains valid if d_L is replaced by the usual Lévy metric.

Now let $(S, \mathcal{F})$ be a PSM space and, for any p, q in S, let

$$\beta(p, q) = d_L(F_{pq}, \epsilon_0). \tag{2.3}$$

It is easy to see that $\beta(p, q) = 0$ if and only if $p = q$ and that $\beta(p, q) = \beta(q, p)$. Thus β is a semimetric on S. As regards the triangle inequality, we have

Theorem 2.1. If $(S, \mathcal{F}, \tau)$ is a PM space with $\tau \geq \tau_W$, then β is a metric on S.

Proof. Suppose $\beta(p, q) = h_1$ and $\beta(q, r) = h_2$. Let $\eta_1, \eta_2 > 0$ be given. Then $\beta(p, q) < h_1 + \eta_1$ and $\beta(q, r) < h_2 + \eta_2$. Therefore we have

$$\begin{aligned}
&\tau(F_{pq}, F_{qr})(h_1 + \eta_1 + h_2 + \eta_2) \\
&\qquad \geq \tau_W(F_{pq}, F_{qr})(h_1 + \eta_1 + h_2 + \eta_2) \\
&\qquad = \sup_{x+y=h_1+\eta_1+h_2+\eta_2} \operatorname{Max}(F_{pq}(x) + F_{qr}(y) - 1, 0) \\
&\qquad \geq F_{pq}(h_1 + \eta_1) + F_{qr}(h_2 + \eta_2) - 1 \\
&\qquad > 1 - h_1 - \eta_1 + 1 - h_2 - \eta_2 - 1 \\
&\qquad = 1 - (h_1 + \eta_1 + h_2 + \eta_2).
\end{aligned}$$

Since $F_{pr} \geq \tau(F_{pq}, F_{qr})$ and $d_L(G, \epsilon_0) \leq d_L(F, \epsilon_0)$ whenever $F \leq G$, we have

$$\begin{aligned}\beta(p,r) = d_L(F_{pr}, \epsilon_0) &\leq d_L(\tau(F_{pq}, F_{qr}), \epsilon_0) \\ &< h_1 + \eta_1 + h_2 + \eta_2 \\ &= \beta(p,q) + \beta(q,r) + \eta_1 + \eta_2.\end{aligned}$$

Letting $\eta_1, \eta_2 \to 0$ completes the proof.

A slightly less general version of Theorem 2.1 is due to Radu [3]. We also note that Hicks [1] called a metric d compatible with $\mathcal{F}$ if $d(p,q) < t$ if and only if $F_{pq}(t) > 1 - t$. Thus β is compatible with $\mathcal{F}$.

It is apparent from the definition of β, (1.2) and (2.1) that f is an H-contraction if and only if, for all $x > 0$ and all p, q in S,

(2.4) $$\beta(fp, fq) < \gamma x \text{ whenever } \beta(p,q) < x;$$

and this observation leads at once to the following result, which is (essentially) due to Radu [3]):

Theorem 2.2. The mapping $f : S \to S$ is an H-contraction on the PM space $(S, \mathcal{F}, \tau)$ with $\tau \geq \tau_M$ if and only if f is a metric contraction on the metric space (S, β), i.e., if and only if there is a $\gamma \in (0,1)$ such that

(2.5) $$\beta(fp, fq) \leq \gamma \beta(p,q) \text{ for all } p, q \in S.$$

It is natural to ask whether the condition $\tau \geq \tau_W$ in Theorem 2.1 is necessary. The general question is open but, as the following example shows, in the class of Menger spaces, the condition is necessary.

Example 2.1. Let T be a (continuous) t-norm and suppose it is not true that $\tau_T \geq \tau_W$. Then it is not true that $T \geq W$. Consequently, there exist a, b with $0 < a \leq b < 1$ such that $0 \leq T(a,b) < W(a,b)$, whence $W(a,b) = a + b - 1$.

Now let $S = \{p, q, r\}$ and define $\mathcal{F} : S \times S \to \Delta^+$ via

$$F_{pq}(x) = \begin{cases} 0, & x \leq 0, \\ a, & 0 < x \leq 2, \\ 1, & 2 < x, \end{cases}$$

$$F_{qr}(x) = \begin{cases} 0, & x \leq 0, \\ b, & 0 < x \leq 2, \\ 1, & 2 < x, \end{cases}$$

and

$$F_{pr}(x) = \tau_T(F_{pq}, F_{qr})(x) = \begin{cases} 0, & x \leq 0, \\ T(a,b), & 0 < x \leq 2, \\ b, & 2 < x \leq 4, \\ 1, & 4 < x, \end{cases}$$

It is tedious but easy to verify that $(S, \mathcal{F}, \tau_T)$ is a PM space. But,

$$\begin{aligned} \beta(p,q) + \beta(q,r) &= d_L(F_{pq}, \epsilon_0) + d_L(F_{qr}, \epsilon_0) \\ &= 1 - a + 1 - b = 1 - W(a,b) \\ &< 1 - T(a,b) = d_L(F_{pr}, \epsilon_0) \\ &= \beta(p,r). \end{aligned}$$

Thus β is not a metric on S.

3. B-contractions and H-contraction compared.

It follows at once from Theorem 2.2 and the fact that (S, β) is complete if and only if $(S, \mathcal{F}, \tau_M)$ is complete, that every H- contraction on a complete PM space $(S, \mathcal{F}, \tau)$ with $\tau \geq \tau_W$ has a unique fixed point (see also [3]). But as pointed out in Section 1, this is not true for B-contractions [7]. Thus a B-contraction need not be an H-contraction. Similarly, as the following example shows, an H-contraction need not be a B-contraction.

Example 3.1. Let $S = \{0, 1, 2, \cdots\}$ and, for $p \neq q$, define $\mathcal{F} : S \times S \to \Delta^+$ via

$$F_{pq}(x) = F_{qp}(x) = \begin{cases} 0, & x \leq 2^{-\mathrm{Min}(p,q)}, \\ 1 - 2^{-\mathrm{Min}(p,q)}, & 2^{-\mathrm{Min}(p,q)} < x \leq 1, \\ 1, & 1 < x. \end{cases}$$

It is straightforward but tedious to verify that $(S, \mathcal{F}, \tau_M)$ is a PM space. Define $f : S \to S$ via $f(r) = r + 1$. Since $\tau_M \geq \tau_W$ and $\beta(fp, fq) = \frac{1}{2}\beta(p, q)$, we have that f is an H-contraction.

Next, let γ be any number in (0,1) and choose $x \in (1, 1/\gamma)$. Then $\gamma x < 1$ so that

$$F_{f(0)f(1)}(\gamma(x)) = F_{12}(\gamma x) \leq 1/2 < 1 = F_{01}(x),$$

whence f is not a B-contraction on $(S, \mathcal{F})$.

It is to be noted that M is the strongest t-norm and that even in this extreme case it is not true that every H-contraction is a B-contraction.

Next, we show that the notion of a B-contraction is sometimes stronger than that of an H-contraction. To this end we need the following:

Lemma 3.1. If f is a B-contraction on the PSM space $(S, \mathcal{F})$ and if the distribution function F_{fpfq} is strictly increasing on [0,1], then $\beta(fp, fq) < \beta(p, q)$.

Proof. Choose η such that $0 < \eta < \dfrac{1-\gamma}{\gamma}\beta(p, q)$. Then $\beta(p, q) > \gamma[\beta(p, q) + \eta]$. Since F_{fpfq} is strictly increasing on [0,1], $0 \leq \beta(p, q) \leq 1$, and f is a B-contraction, we have

$$\begin{aligned} F_{fpfq}(\beta(p, q)) &> F_{fpfq}(\gamma[\beta(p, q) + \eta]) \geq F_{pq}(\beta(p, q) + n) \\ &\geq F_{pq}(\beta(p, q)+) > 1 - \beta(p, q), \end{aligned}$$

where the last inequality follows from (2.2). Thus, again by (2.2), $\beta(fp, fq) < \beta(p, q)$.

Theorem 3.1. Let $(S, \mathcal{F})$ be a PSM space. Suppose that $\mathrm{Ran}(\mathcal{F})$ is finite and that each element of $\mathrm{Ran}(\mathcal{F})\setminus\{\epsilon_0\}$ is strictly increasing on [0,1]. Then every B-contraction on $(S, \mathcal{F})$ is an H-contraction.

Proof. Suppose f is a B-contraction on $(S, \mathcal{F})$. Then Lemma 3.1 implies that, for any pair of points p, q in S, there exists a $\gamma_{pq} \in (0, 1)$ such that $\beta(fp, fq) < \gamma_{pq}\beta(p, q)$. Since $\mathrm{Ran}(\mathcal{F})$ is finite, there is a $\gamma \in (0, 1)$ such that $\mathrm{Max}\{\gamma_{pq} | p, q \in S\} < \gamma < 1$. Thus $\beta(fp, fq) \leq \gamma\beta(p, q)$ for all p, q in S, whence by Theorem 2.2, f is an H- contraction.

Note that neither the definition of an H-contraction nor the definition of d_L involves values of the distribution functions outside of the interval [0,1] (see (1.2) and (2.2)). Thus, in Lemma 3.1 and Theorem 3.1, it is only necessary to assume that the distribution functions in question are strictly increasing on [0,1].

In general, it is not true that every B-contraction is an H-contraction. To show this, we give two examples. The first shows that in Theorem 3.1 we cannot get rid of that condition that each $F_{pq} \neq \epsilon_0$ is strictly increasing on [0,1], while the second shows that we cannot get rid of the condition that $\mathrm{Ran}(\mathcal{F})$ is finite.

Example 3.2. Let $S = \{p, q, r\}$ and let $\mathcal{F} : S \times S \to \Delta^+$ be defined via

$$F_{pr}(x) = F_{rp}(x) = F_{rq}(x) = F_{qr}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1/2, & \text{if } 0 < x \leq 2, \\ 1, & \text{if } x > 2, \end{cases}$$

and

$$F_{pq}(x) = F_{qp}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1/2, & \text{if } 0 < x \leq 3/2, \\ 1, & \text{if } x > 3/2. \end{cases}$$

It is again tedious but straightforward to show that $(S, \mathcal{F}, \tau_M)$ is a PM space. Define f as follows: $f(p) = f(q) = p$ and $f(r) = q$. Since $F_{pq}(3x/4) = F_{pr}(x)$ for all x, it follows at once that f is a B- contraction on $(S, \mathcal{F})$. Theorem 2.2 shows, however, that f is not an H-contraction because $\beta(fp, fr) = \beta(p, q) = \dfrac{1}{2} = \beta(p, r)$.

Example 3.3. For each integer n, let $p_n : (0,1) \to R^+$ be given by $p_n(t) = 2^{-n}(1-t)t^{-1}$. Let $S = \{p_n : n \text{ is an integer }\}$. Let P be Lebesgue measure on (0,1), and for $x \geq 0$ let

$$\begin{aligned} F_{p_n p_m}(x) &= P\{t \in (0,1) : |p_n(t) - p_m(t)| < x\} \\ &= \frac{x}{x + |2^{-n} - 2^{-m}|}. \end{aligned}$$

The PSM $(S, \mathcal{F})$ is an E-space [5], which automatically yields that $(S, \mathcal{F}, \tau_W)$ is a PM space. Mores is true, however, since it can readily be shown that $(S, \mathcal{F}, \tau_M)$ is a PM space.

Let $f : S \to S$ be defined by $f(p_n) = p_{n+1}$. Since

$$F_{fp_n fp_m}\left(\frac{1}{2}x\right) = F_{p_{n+1}p_{m+1}}\left(\frac{1}{2}x\right) = F_{p_n p_m}(x),$$

f is a B-contraction.

Now suppose f is an H-contraction. Then there is some γ in (0,1) such that

$$\beta(f_{p_n}, f_{p_m}) < \gamma\beta(p_n, p_m)$$

for all integers n and m. It is easy to calculate that

$$\beta(p_n, p_m) = \left(\sqrt{|2^{-n} - 2^{-m}|^2 + 4|2^{-n} - 2^{-m}|} - |2^{-n} - 2^{-m}|\right)/2$$

and that

$$\beta(fp_n, fp_m) = \left(\sqrt{|2^{-n-1} - 2^{-m-1}|^2 + 4|2^{-n-1} - 2^{-m-1}|} - |2^{-n-1} - 2^{-m-1}|\right)/2.$$

Hence, we must have

$$\gamma > \frac{\sqrt{|2^{-n-1} - 2^{-m-1}|^2 + 4|2^{-n-1} - 2^{-m-1}|} - |2^{-n-1} - 2^{-m-1}|}{\sqrt{|2^{-n} - 2^{-m}|^2 + 4|2^{-n} - 2^{-m}|} - |2^{-n} - 2^{-m}|}.$$

Taking the limit as $m \to \infty$, we obtain

$$\gamma > \frac{\sqrt{|2^{-2n-2} + 2^{-n+1}} - 2^{-n-1}}{\sqrt{|2^{-2n} + 2^{-n+2}} - 2^{-n}} = \frac{\sqrt{1 + 2^{n+2}} + 1}{\sqrt{1 + 2^{n+3}} + 1}.$$

Finally, taking the limit as $n \to -\infty$ yields $\gamma \geq 1$ which is a contradiction. Thus f is not an H-contraction.

As is well-known, E-spaces are pseudometrically generated spaces, and conversely [6]. In Example 3.3, the generating pseudometrics are given by

$$\delta_t(p_n, p_m) = |p_n(t) - p_m(t)| = |2^{-n} - 2^{-m}|\frac{(1-t)}{t}, \text{ for } t \in (0,1).$$

Now observe that

$$\begin{aligned}\delta_t(f(p_n), f(p_m)) &= |p_{n+1}(t) - p_{m+1}(t)| \\ &= \frac{1}{2}|p_n(t) - p_m(t)| = \frac{1}{2}\delta_t(p_n, p_m).\end{aligned}$$

Thus f is a contraction on each of the pseudometric spaces (S, δ_t). Nevertheless, f is not an H-contraction!.

Note that in both examples we have Menger spaces under M. Thus, even in the presence of this strong version of the triangle inequality, B-contractions need not be H-contractions.

4. Contractions on α-simple spaces.

Theorem 4.1. For $\alpha > 0$, let (S, G, d, α) be the α- simple space generated by the strictly increasing distribution function G in Δ^+ and the metric space (S, d). Then f is a B-contraction on (S, G, d, α) if and only if f is a contraction on (S, d).

Proof. Let $\gamma \in (0,1)$ and $x > 0$ be given. Then

$$G\left(\frac{\gamma x}{d(fp, fq)^\alpha}\right) = F_{fpfq}(\gamma x) \geq F_{pq}(x) = G\left(\frac{x}{d(p,q)^\alpha}\right),$$

which is true if and only if

$$d(fp, fq) \leq \gamma^{1/\alpha} d(p,q).$$

For H-contractions, the situation is different. Here we first have,

Theorem 4.2. Let G in Δ^+ be strictly increasing and let (S, G, d, α) be the α-simple space generated by G and (S, d). Then every H-contraction on (S, G, d, α) is a contraction on (S, d).

Proof. If F_{pq} is strictly increasing, then its quasi-inverse $F_{pq}^\wedge$ is continuous; and since the graphs of j and $F_{pq}^\wedge(1-j)$ have a unique point of intersection, $\beta(p,q)$ is the unique solution of the equation $x = F_{pq}^\wedge(1-x)$, i.e., $\beta(p,q) = F_{pq}^\wedge(1-\beta(p,q))$. Moreover, if $F_{pq}(x) = G(x/d(p,q)^\alpha)$, then $F_{pq}^\wedge(x) = d(p,q)^\alpha G^\wedge(x)$ and $\beta(p,q) = d(p,q)^\alpha G^\wedge(1-\beta(p,q))$. Thus, if f is an H-contraction, then

$$\begin{aligned} d(fp, fq)^\alpha &= \frac{\beta(fp,fq)}{G^\wedge(1-\beta(fp,fq))} \leq \frac{\gamma\beta(p,q)}{G^\wedge(1-\beta(fp,fq))} \\ &\leq \frac{\gamma\beta(p,q)}{G^\wedge(1-\beta(fp,fq))} = \gamma d(p,q)^\alpha, \end{aligned}$$

since $\beta(fp, fq) \leq \gamma\beta(p, q) < \beta(p, q)$. Consequently,

$$d(fp, fq) \leq \gamma^{1/\alpha} d(p, q).$$

The converse of Theorem 4.2 is false, as the following example shows.

Example 4.1. Let $G(x) = 0$ for $x \leq 0$ and $G(x) = x/(x+1)$ if $x > 0$, and let (S, d) be the Euclidean line. Then $(S, G, d, 1)$ is a simple space. Define $f : S \to S$ via $f(p) = \frac{1}{2}p$. Clearly f is a contraction on (S, d). However, f is not an H-contraction on $(S, G, d, 1)$. To see this, suppose tho the contrary that there is a $\gamma \in (0, 1)$ such that (1.2) holds. Then elementary calculations yield that for all p, q in S and x in (0,1),

$$F_{pq}(x) > 1 - x \text{ if and only if } |p - q| < \frac{x^2}{1 - x}$$

and, consequently,

$$F_{fpfq}(\gamma x) > 1 - \gamma x \text{ if and only if } |p - q| < \frac{2\gamma^2 x^2}{1 - \gamma x}.$$

Since (1.2) holds, it follows that

$$|p - q| < \frac{x^2}{1 - x} \text{ implies } |p - q| < \frac{2\gamma^2 x^2}{1 - \gamma x}.$$

for all p, q in S and x in (0,1). From (4.1) it follows that

$$\frac{x^2}{1 - x} \leq \frac{2\gamma^2 x^2}{1 - \gamma x}$$

or, equivalenty,

$$1 - \gamma x \leq 2\gamma^2(1 - x).$$

Letting x tend to 1 from below yields $\gamma \geq 1$, a contradiction.

5. Concluding remarks.

(1) Let $(S, \mathcal{F})$ be a pseudometrically generated PM space, f a mapping from S into S, and γ a fixed number in (0,1). In [7] it was shown that if f is a strict contraction on $(S, \mathcal{F})$, i.e., if f is a contraction mapping with contraction constant γ on each of the pseudometric spaces that generate $(S, \mathcal{F})$, then f is a B- contraction. However, as the discussion following Example 3.2 shows, a strict contraction need not be an H-contraction.

(2) Let $(S, \mathcal{F}, \tau_M)$ be a Menger space under Min and, for any p, q in S and c in (0,1), define $d_c : S \times S \to R^+$ via

$$d_c(p, q) = F_{pq}^{\frown}(c),$$

where $F_{pq}^{\frown}$ is the left-continuous quasi-inverse of F_{pq}. Then (see [4, Section 12.6]) each of the functions d_c is a pseudometric on S; and if f is a B-contraction on $(S, \mathcal{F}, \tau_M)$, then f is a contraction on each of the pseudometric spaces (S, d_c).

In the particular case of Example 3.2, we have that

$$F_{p_n p_m}^{\frown}(c) = |2^{-n} - 2^{-m}| \frac{c}{1-c},$$

whence, in view of (3.1), the quasi-inverse pseudometrics $\{d_c\}$ and the generating pseudometrics $\{\delta_t\}$ are related by $d_c = \delta_{1-c}$. Thus again the mapping f in Example 3.2 is a contraction on each of the pseudometric spaces (S, d_c), but not an H-contraction.

(3) Clearly, from the probabilistic point of view, B-contractions and H- contractions both have their shortcomings. It would be desirable to have a notion of a contraction on a PM space which has the following properties:

(a) If the PM space $(S, \mathcal{F}, \tau)$ is complete and $\tau \geq \tau_W$, then any contraction on S has a unique fixed point (at least with a high probability).

(b) If $(S, \mathcal{F})$ is a pseudometrically generated PM space, then every strict contraction on S is a contraction.

Note again that B-contractions do not satisfy (a) and H-contractions do not satisfy (b).

(4) As Theorem 2.2 shows, H-contractions are metric contractions. Similarly, in all instances in which they have unique fixed points, B-contractions on PM spaces are metric contractions on related pseudometric spaces. This is so for Menger spaces under Min (see (2) above) and also for Menger spaces under t-norms that are equicontinuous at the point (1.1) [2].

(5) Since the appearance of the original paper by Sehgal and Bharucha-Reid, many papers dealing with fixed point theorems on PM spaces have been published. Of these, by far the vast majority deal with Menger spaces under Min or very closely related t-norms (see (4) above). Since contractions on these spaces are contractions on associated pseudometric spaces, it should come as no surprise that known fixed point theorems for ordinary metric spaces can be "generalized". Unfortunately, most of the authors of the papers in question appear to be unaware of this situation. It is also quite remarkable that, to the best of our combined knowledge, not one of these many papers contains even a single example or counterexample!

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