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FAITHFUL HOMOGENEOUS SPACES OVER COMMUTATIVE GROUPS AND TST-SPACES

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ABSTRACT

A set Q is a faithful homogeneous space over a commutative group iff there is a family S of mappings such that (Q,S) is a TST-space.

If a group $(G, \circ)$ of mappings acts strictly transitively on a given set X, then X is said to be a faithful homogeneous space over the group $(G, \circ)$, in accordance with [1].

In [2] the notion of a parallelogram space on a given set is defined and it is proved that there is the structure of a parallelogram space on a given set Q iff Q is a faithful homogeneous space over a commutative group.

In [3] the notion of a TST-space is defined (see [4]). A TST-space (Q, S) is a nonvoid set Q with a family S of its involutory mappings (which are called symmetries) such that S acts transitively on Q and from $\sigma_1, \sigma_2, \sigma_3 \in S$ it follows $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$, where $\circ$ is the composition of mappings. In the same paper [3] it is proved that there is the structure of a parallelogram space on a given set Q iff there is a family S of its mappings such that (Q, S) is a TST-space.

From these two results it follows immediately the following theorem.

Theorem. A set Q is a faithful homogeneous space over a commutative group $(T, \circ)$ iff there is a family S of its mappings such that (Q, S) is a TST-space.

But, we shall give here a direct proof of this theorem.

Let Q be a faithful homogeneous space over a commutative group $(T,\circ)$, i.e. let this group act strictly transitively on the set Q.

For any $a,b\in Q$ let us denote by $\tau_{a,b}$ the element of T such that $\tau_{a,b}(a)=b$. Then we have identities

$$\tau_{a,b}^{-1}=\tau_{b,a},$$

$$\tau_{a,b}\circ\tau_{b,c}=\tau_{b,c}\circ\tau_{a,b}=\tau_{a,c}$$

and for any $a\in Q$ the mapping $\tau_{a,a}$ is the identity. Now, for any $a,b\in Q$ we define a symmetry $\sigma_{a,b}:Q\to Q$ by the equivalence

$$\sigma_{a,b}(c)=d\iff\tau_{a,c}(d)=b, \tag{1}$$

which can be written in the form

$$\sigma_{a,b}(c)=d\iff\tau_{a,c}=\tau_{d,b}. \tag{2}$$

From $\sigma_{a,b}(c)=d$, i.e. $\tau_{a,c}=\tau_{d,b}$, it follows by the commutativity of the group $(T,\circ)$ successively

$$(\tau_{d,b}\circ\tau_{c,b})(a)=(\tau_{a,c}\circ\tau_{c,b})(a)=(\tau_{c,b}\circ\tau_{a,c})(a)=\tau_{c,b}(c)=b,$$

i.e.

$$\tau_{c,b}(a)=\tau_{d,b}^{-1}(b)=d,$$

or $\tau_{c,b}=\tau_{a,d}$. But, $\tau_{a,d}=\tau_{c,b}$ implies by (2) $\sigma_{a,b}(d)=c$. Therefore, from $\sigma_{a,b}(c)=d$ it follows $\sigma_{a,b}(d)=c$ and every symmetry is an involutory mapping. Because $\tau_{a,a}=\tau_{b,b}$ (the identity), from (2) we conclude that $\sigma_{a,b}(a)=b$ and the family S of all symmetries acts transitively on the set Q. We shall show that this action is strict. Indeed, let $a,b\in Q$ be any elements. It suffices to show that $\sigma_{c,d}(a)=b$ implies $\sigma_{c,d}=\sigma_{a,b}$, i.e. that from

$\sigma_{c,d}(x) = y$ it follows $\sigma_{a,b}(x) = y$. According to (1) we must prove that from $\tau_{c,a}(b) = d$ and $\tau_{c,x}(y) = d$ it follows $\tau_{a,x}(y) = b$. But, we obtain successively

$$\tau_{a,x}(y) = (\tau_{a,c} \circ \tau_{c,x})(y) = \tau_{a,c}(d) = \tau_{c,a}^{-1}(d) = b.$$

Therefore, for any $a, b \in Q$ and any $\sigma \in S$ from $\sigma(a) = b$ it follows $\sigma = \sigma_{a,b}$. Now, let $\sigma_1, \sigma_2, \sigma_3 \in S$ be any symmetries. Let $a \in Q$ be any element and $b = \sigma_1(a)$, $c = \sigma_2(b)$, $d = \sigma_3(c)$. Then $\sigma_1 = \sigma_{a,b}$, $\sigma_2 = \sigma_{b,c}$, $\sigma_3 = \sigma_{c,d}$. We shall prove the equalities

$$\sigma_{b,c} \circ \sigma_{a,b} = \tau_{a,c}, \quad \sigma_{d,c} \circ \sigma_{a,d} = \tau_{a,c}. \tag{3}$$

Let $x \in Q$ be any element and $y = \sigma_{a,b}(x)$, $z = \sigma_{b,c}(y)$. Then we have by (2) $\tau_{a,x} = \tau_{y,b}$, $\tau_{b,y} = \tau_{z,c}$ and it follows

$$\tau_{a,c} = \tau_{a,x} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{b,y}^{-1} \circ \tau_{x,c} = \tau_{z,c}^{-1} \circ \tau_{x,c} = \tau_{c,z} \circ \tau_{x,c} = \tau_{x,z},$$

i.e.

$$\tau_{a,c}(x) = z = \sigma_{b,c}(y) = (\sigma_{b,c} \circ \sigma_{a,b})(x).$$

Therefore, we have the first equality (3). The second equality (3) can be proved analogously. But, from (3) it follows

$$\sigma_{a,d} = \sigma_{d,c}^{-1} \circ \sigma_{b,c} \circ \sigma_{a,b} = \sigma_{c,d} \circ \sigma_{b,c} \circ \sigma_{a,b} = \sigma_3 \circ \sigma_2 \circ \sigma_1,$$

i.e. $\sigma_3 \circ \sigma_2 \circ \sigma_1$ is a symmetry. We have proved the following proposition.

Proposition 1. If Q is a faithful homogeneous space over a commutative group $(T, \circ)$ and if S is the set of all symmetries defined by (1), then (Q, S) is a TST-space.

Now, let (Q, S) be any TST-space, i.e. let S be a family of involutory mappings (symmetries) of the set Q which acts transitively on Q and such that $\sigma_1, \sigma_2, \sigma_3 \in S$ implies $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$.

The involutority of symmetries implies obviously their bijectivity. Let T be the family of all compositions of two symmetries. If $\sigma_2 \circ \sigma_1$, $\sigma_4 \circ \sigma_3 \in T$, then

$$(\sigma_4 \circ \sigma_3) \circ (\sigma_2 \circ \sigma_1)^{-1} = \sigma_4 \circ \sigma_3 \circ \sigma_1 \circ \sigma_2 \in T.$$

because of $\sigma_3 \circ \sigma_1 \circ \sigma_2 \in S$. Therefore, $(T, \circ)$ is a subgroup of the group generated by all symmetries. Moreover, we have

$$(\sigma_4 \circ \sigma_3) \circ (\sigma_2 \circ \sigma_1) = \sigma_4 \circ (\sigma_3 \circ \sigma_2 \circ \sigma_1)^{-1} = \sigma_4 \circ \sigma_1 \circ \sigma_2 \circ \sigma_3$$

$$= (\sigma_4 \circ \sigma_1 \circ \sigma_2)^{-1} \circ \sigma_3 = (\sigma_2 \circ \sigma_1) \circ (\sigma_4 \circ \sigma_3),$$

and $(T, \circ)$ is a commutative group. In [3] (see [4]) it is proved that the family S acts strictly transitively on Q, i.e. that for any $\sigma_1, \sigma_2 \in S$ and any $a \in Q$ from $\sigma_1(a) = \sigma_2(a)$ it follows $\sigma_1 = \sigma_2$. Then transitivity of T on Q follows immediately from the transitivity of S on Q. But, in [3] it is proved that T acts strictly transitively on Q. Indeed, let $\tau_1(a) = \tau_2(a)$ for any $\tau_1, \tau_2 \in T$ and any $a \in Q$. Then, for some $\sigma \in S$ we have $(\sigma \circ \tau_1)(a) = (\sigma \circ \tau_2)(a)$. But, $\sigma \circ \tau_1, \sigma \circ \tau_2 \in S$ and hence $\sigma \circ \tau_1 = \sigma \circ \tau_2$, i.e. $\tau_1 = \tau_2$. Therefore, Q is a faithful homogeneous space over the group $(T, \circ)$. For any $a, b \in Q$ let $\sigma_{a,b} \in S$ resp. $\tau_{a,b} \in T$ be such elements that $\sigma_{a,b}(a) = a$ resp. $\tau_{a,b}(a) = b$. Then for any $a, b, c \in Q$ we obtain

$$(\sigma_{b,c} \circ \sigma_{a,b})(a) = \sigma_{b,c}(b) = c = \tau_{a,c}(a),$$

i.e. $\sigma_{b,c} \circ \sigma_{a,b} = \tau_{a,c}$. Let us prove the equivalence (1). Let $\sigma_{a,b}(c) = d$. Then we have $\sigma_{a,b}(d) = c$ and from $\sigma_{b,c}(b) = c$ it follows $\sigma_{b,c}(c) = b$. Therefore, we get

$$\tau_{a,c}(d) = (\sigma_{b,c} \circ \sigma_{a,b})(d) = \sigma_{b,c}(c) = b.$$

Thus, we have proved the following proposition.

Proposition 2. If (Q, S) is a TST-space and T the family of all compositions of two elements of S, then $(T, \circ)$ is a commutative group, Q is a faithful homogeneous space over it and the equivalence (1) holds.

Propositions 1 and 2 prove our Theorem.

References.

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