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LATERALLY COMMUTATIVE HEAPS AND TST-SPACES

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ABSTRACT

A laterally commutative heap can be defined on a given set iff there is the structure of a TST-space on this set.

Let $(\ ) : Q^3 \to Q$ be a ternary operation on a set Q. In the terminology of [3] $(Q, (\ ))$ is said to be a laterally commutative heap if the following identities hold:

(1) $$(abc) = (cba),$$

(2) $$((abc)de) = (a(bcd)e),$$

(3) $$(abb) = a.$$

On the basis of the results of [4] it follows that $(Q, (\ ))$ is a laterally commutative heap iff a prallelogram space can be defined on the set Q, whose notion is defined in [1] (see [2]).

Let S be a family of mappings of the set Q. In the terminology of [2] (Q, S) is said to be a TST-space if the following properties are satisfied ($\circ$ is the composition of mappings):

(S1) For every $\sigma \in S$ the mapping $\sigma \circ \sigma$ is the identity.

(S2) For any $a, b \in Q$ there is a $\sigma \in S$ such that $\sigma(a) = b$.

(S3) From $\sigma_1, \sigma_2, \sigma_3 \in S$ it follows $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$.

In [2] it is proved that (Q, S) is a TST-space iff a parallelogram space can be defined on the set Q.

Therefore, the following theorem is valid.

Theorem. There is a laterally commutative heap $(Q, (\ ))$ iff there is a TST-space (Q, S).

But, we shall give a direct proof of this theorem. It suffices to prove the following two propositions.

Proposition 1. Let $(Q, (\ ))$ be a laterally commutative heap and for any $a, b \in Q$ let $\sigma_{a,b} : Q \to Q$ be the mapping defined by

$$\sigma_{a,b}(x) = (axb). \tag{4}$$

If S is the set of all mappings of the form $\sigma_{a,b}$, then (Q, S) is a TST-space.

Proof. For every $x \in Q$ we have

$$(\sigma_{a,b} \circ \sigma_{a,b})(x) =^{(4)} (a(axb)b)$$

$$=^{(2)} ((aax)bb) =^{(3)} (aax) =^{(1)} (xaa) =^{(3)} x$$

and $\sigma_{a,b} \circ \sigma_{a,b}$ is the identity. For any $a, b \in Q$ we obtain

$$\sigma_{a,b}(a) =^{(4)} (aab) =^{(1)} (baa) =^{(3)} b.$$

For any $\sigma_{a,b}, \sigma_{c,d}, \sigma_{e,f} \in S$ and every $x \in Q$ we get successively

$$(\sigma_{e,f} \circ \sigma_{c,d} \circ \sigma_{a,b})(x) =^{(4)} (e(c(axb)d)f)$$

$$=^{(2)} (e((cax)bd)f) =^{(2)} ((e(cax)b)df) =^{(2)} (((eca)xb)df)$$

$$=^{(1)} (fd(bx(ace))) =^{(2)} ((fdb)x(ace)) =^{(4)} \sigma_{(fdb),(ace)}(x),$$

i.e.,

$$\sigma_{e,f} \circ \sigma_{c,d} \circ \sigma_{a,b} = \sigma_{(fdb),(ace)} \in S.$$

Now, let (Q, S) be a TST-space. Let us prove (as in [2]) that for any $\sigma_1, \sigma_2 \in S$ and any $a \in Q$ from $\sigma_1(a) = \sigma_2(a)$ it follows $\sigma_1 = \sigma_2$. From $\sigma_1(a) = \sigma_2(a) = b$, because of (S1), it follows $\sigma_2(b) = a$ and hence $(\sigma_2 \circ \sigma_1)(a) = a$. Now, let $x \in Q$ be any element. By (S2) there is a $\sigma \in S$ such that $\sigma(a) = x$. According to (S3) we have $\sigma \circ \sigma_2 \circ \sigma_1 \in S$ and then we obtain by (S1) successively

$$\sigma_1(x) = (\sigma_1 \circ \sigma)(a) = (\sigma_1 \circ \sigma \circ \sigma_2 \circ \sigma_1)(a) = [\sigma_1 \circ (\sigma \circ \sigma_2 \circ \sigma_1)^{-1}](a) =$$

$$= (\sigma_1 \circ \sigma_1 \circ \sigma_2 \circ \sigma)(a) = (\sigma_2 \circ \sigma)(a) = \sigma_2(x),$$

i.e., $\sigma_1 = \sigma_2$. Therefore, together with (S2), we conclude that for any $a, b \in Q$ there is a unique $\sigma \in S$ such that $\sigma(a) = b$. Let us denote this σ by $\sigma_{a,b}$. For any $a, b \in Q$ it follows by (S1)

$$\sigma_{a,b} = \sigma_{b,a}, \tag{5}$$

and for any $a, b, c, d \in Q$ we have, according to (S3), the equality

$$\sigma_{c,d} \circ \sigma_{b,c} \circ \sigma_{a,b} = \sigma_{a,d}. \tag{6}$$

Therefore, for any $a, b, c, d, e \in Q$ we have

$$\sigma_{b,e} \circ \sigma_{c,b} \circ \sigma_{a,c} = \sigma_{a,e} = \sigma_{b,e} \circ \sigma_{d,b} \circ \sigma_{a,d},$$

wherefrom it follows the identity

$$\sigma_{c,b} \circ \sigma_{a,c} = \sigma_{d,b} \circ \sigma_{a,d}. \tag{7}$$

Proposition 2. If (Q, S) is a TST-space and $(\ \ ) : Q^3 \to Q$ a ternary operation defined by

$$(abc) = \sigma_{a,c}(b), \tag{8}$$

then $(Q, (\ \ ))$ is a laterally commutative heap.

Proof. We have successively the identities

$$(abc) =^{(8)} \sigma_{a,c}(b) =^{(5)} \sigma_{c,a}(b) =^{(8)} (cba),$$

$$(abb) =^{(8)} \sigma_{a,b}(b) =^{(5)} \sigma_{b,a}(b) = a,$$

i.e., (1) and (3). Let us prove the identity (2). Let $a, b, c, d, e \in Q$ be any elements and let

$$f = \sigma_{b,d}(c), \quad g = \sigma_{a,e}(f), \quad h = \sigma_{a,c}(b), \tag{9}$$

i.e., by (5)

$$\sigma_{d,b} = \sigma_{b,d} = \sigma_{c,f}, \quad \sigma_{a,e} = \sigma_{f,g}, \quad \sigma_{c,a} = \sigma_{a,c} = \sigma_{b,h} = \sigma_{h,b}. \tag{10}$$

Now, we have successively

$$\sigma_{h,e} =^{(6)} \sigma_{a,e} \circ \sigma_{c,a} \circ \sigma_{h,c} =^{(10),(5)} \sigma_{f,g} \circ \sigma_{h,b} \circ \sigma_{c,h}$$

$$=^{(7)} \sigma_{f,g} \circ \sigma_{d,b} \circ \sigma_{c,d} =^{(10),(5)} \sigma_{f,g} \circ \sigma_{c,f} \circ \sigma_{d,c} =^{(6)} \sigma_{d,g},$$

i.e., $\sigma_{h,e}(d) = g$. Therefore, we obtain

$$(a(bcd)e) =^{(8)} \sigma_{a,e}(\sigma_{b,d}(c)) =^{(9)} \sigma_{a,e}(f) =^{(9)} g$$

$$= \sigma_{h,e}(d) =^{(9)} \sigma_{\sigma_{a,c}(b),e}(d) =^{(8)} ((abc)de).$$

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