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EXPLICIT SOLUTIONS FOR STURM-LIOUVILLE OPERATOR PROBLEMS II

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ABSTRACT

It is proved that the resolution problem of an operator Sturm-Liouville operator problem for a second-order differential operator equation with constant coefficients is solved in terms of solutions of the corresponding algebraic operator equation. Existence and uniqueness conditions for the existence on nontrivial solutions of the problem and an explicit expression of them are given.

1. Introduction.

For the scalar case, the solution of boundary value problems for linear partial differential equations may sometimes be reduced to the solution of ordinary differential equations containing a parameter and subject to certain boundary value conditions. So, the classical Sturm-Liouville theory yields a complete solution of the problem

$$X^{(2)} + A_1 X^{(1)} + (A_0 - \lambda I)X = 0 \tag{1.1}$$

$$M_{11}X(a) + N_{11}X(b) + M_{12}X^{(1)}(b) + N_{12}X^{(1)}(b) = 0$$

$$M_{21}X(a) + N_{21}X(b) + M_{22}X^{(1)}(a) + N_{22}X^{(1)}(b) = 0$$

where A_0, A_1, λ, N_{ij}, and M_{ij}, for $1 \leq i,j \leq 2$, are complex numbers and $t \in [a,b]$, see [3], [6], for details.

For the finite-dimensional case, second order operator differential equations are important in the theory of dampde oscillatory systems and vibrational systems, [5]. Infinite-dimensional differential equations occur frequently in the theory of stochastic processes, the degradation of polymers, infinite ladder network theory in engineering, [1], [20], denumerable markov chains, [7]. Infinite-dimensional second order differential equations of the type arising in (1.1) arise in the linear theory of small oscillations of a continuum, [9], [10], and are studied in [4], [8], [11]. Sturm-Liouville operator problems are studied with different techniques in [14], [15], [17], [18], [19], [21], [23].

In order to solve the operator differential equation

$$X^{(2)} + A_1 X^{(1)} + (A_0 - \lambda I)X = 0, \tag{2.1}$$

and in a way analogous to the scalar case, we obtain a fundamental set of solutions of the equation (2.1) from the existence of solutions of the algebraic characteristic equation

$$X^2 + A_1 X + (A_0 - \lambda I) = 0 \tag{3.1}$$

Thus an explicit expression for any solution of (2.1) is given in terms of a pair of solutions of the algebraic equation (3.1). In this sense, this paper may be regarded as a continuation of [12], [13]. Note that for the operator case, the equation (3.1) may be unsolvable; for example, if $A_1 = 0$ and $A_0 - \lambda I$, is a unilateral weighted shift operator on a complex separable Hilbert space H, then the corresponding equation (3.1) is unsolvable, [23], p. 63.

The resolution problem of the equation (3.1) is related to the problem of the existence of a linear factorization of the polynomial operator $L(z) = z^2 + A_1 z + A_0 - \lambda I$. So, for the finite-dimensional case, the equation (3.1) is solvable, if $L(z)$ is linearly factorizable. Several recent characterizations about the problem of the factorization of the polynomial operator $L(z)$ may be found in [5], [16] and [22]. So, for example, if H is a finite-dimensional, (3.1) is solvable if the companion matrix of $L(z)$ is diagonable, [5].

A methodology for solving the algebraic operator equation (3.1) when H is infinite-dimensional is given in [12], by means of the application annihilating operator functions.

This paper is concerned with the study of the eigenvalue operator problem (1.1) where A_0, A_1, N_{ij} and M_{ij}, for $1 \leq i,j \leq 2$, are bounded linear operators on a complex separable Hilbert space H, and λ is a complex parameter. Existence and uniqueness condition for the existence of nontrivial solutions of (1.1) are given and computable expressions for the solutions of (1.1) for the finite-dimensional case are obtained. A particular case of the problem (1.1) for the operator case is studied in [13], where the problem

$$X^{(2)} - \lambda Q X = 0 \tag{4.1}$$

$$E_1 X(0) + E_2 X^{(1)}(0) = 0$$

$$F_1 X(b) + F_2 X^{(1)}(b) = 0$$

is treated from the same point of view. Note that for the problem (4.1), the solvability of the corresponding algebraic equation $X^2 - \lambda Q = 0$, means that λQ has a square root, and in this case an explicit expression of the solutions of this equation is available by means of the Riesz-Dunford functional calculus, see [13], for details.

Throughout this paper H will denote a complex separable Hilbert space and $L(H)$ will denote the algebra of all bounded linear operators on H. If T lies in $L(H)$, its spectrum $\sigma(T)$ is the set of all complex numbers z such that $zI - T$ is invertible in $L(H)$ and its compression spectrum $\sigma_{\text{comp}}(T)$ is the set of all complex numers z such that the range $(zI - T)(H)$ is not dense in H, [2], pp. 240.

2. Sturm-Liouville operator problems for the equation $X^{(2)} + A_1 X^{(1)} + A_0 - \lambda I)X = 0$.

If we consider the algebra $L(H)$ with the strong operator topology, we obtain a topological vector space which will be denoted by $L_S(H)$. In either one of the two spaces $L_S(H)$ or $L(H)$ we can look the operator differential equation (1.1). If we consider the operator differential equation (2.1) on an interval J, we say that X is a solution of (2.1) on J, if at each point t of J, there exist the strong derivatives $X^{(i)}(t)$, for $i = 1, 2$, and the equation (2.1) is satisfied by X for all t in the interval J.

Theorem 2.1. (i) Let X_0 be a double root of the equation (3.1), that is, a solution of (3.1) such that $2X_0 + A_1 = 0$, then any solution X of (2.1) on the interval $[a, b]$, may be expressed in the form

$$X(t) = \exp(tX_0)(T_1 + tT_2) \tag{1.2}$$

where the operators T_1 and T_2 are given by the expressions $C_0 = X(a)$, $X^{(1)}(a) = C_1$ and

$$T_1 = \exp(-aX_0)((aX_0 + I)C_0 - aC_1)); \quad T_2 = \exp(-aX_0)(C_1 - X_0C_0) \tag{2.2}$$

(ii) Let X_0, X_1 be two solutions of the equation (3.1) such that $X_1 - X_0$ is invertible in $L(H)$, then any solution X of (2.1) on the interval $[a, b]$, may be expressed in the form

$$X(t) = \exp(tX_0)T_1 + \exp(tX_1)T_2 \tag{3.2}$$

where $C_0 = X(a)$, $X^{(1)}(a) = C_1$ and

$$T_1 = \exp(-aX_0)(I + (X_1 - X_0)^{-1}C_0 - (X_1 - X_0)^{-1}C_1; \tag{4.2}$$

$$T_2 = \exp(-aX_1)(-(X_1 - X_0)^{-1}X_0C_0 + (X_1 - X_0)^{-1}C_1)$$

Proof. (i) It is clear that under the hypothesis, the operator functions $U_1(t) = \exp(tX_0)$ and $U_2(t) = t\exp(tX_0)$, are solutions of the equation of (2.1). Let X be a solution of the equation (2.1) on the interval $[a, b]$, and let $C_0 = X(a)$, $C_1 = X^{(1)}(a)$. If we consider the operator function $U(t) = \exp(tX_0)(T_1 + tT_2)$, where T_1 and T_2 are unknown operators in $L(H)$, in order to satisfy the Cauchy problem

$$Y^{(2)} + A_1Y^{(1)} + (A_0 - \lambda)Y = 0; \quad Y(a) = C_0, \quad Y^{(1)}(a) = C_1 \tag{5.2}$$

the operators T_1 and T_2 must verify the following conditions

$$U(a) = \exp(aX_0)(T_1 + aT_2) = C_0 \tag{6.2}$$

$$U^{(1)}(a) = \exp(aX_0)(T_2 + X_0(T_1 + aT_2))$$

The system (6.2) is equivalent to the system

$$\begin{bmatrix} \exp(aX_0) & a\exp(aX_0) \\ X_0\exp(aX_0) & (aX_0 + I)\exp(aX_0) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \tag{7.2}$$

$$\begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_0) \end{bmatrix} \begin{bmatrix} I & aI \\ X_0 & aX_0 + I \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \tag{8.2}$$

From lemma 1 of [13], it follows that

$$\begin{bmatrix} T & aI \\ X_0 & aX_0 + I \end{bmatrix}^{-1} = \begin{bmatrix} aX_0 + I & -aI \\ -X_0 & I \end{bmatrix} \tag{9.2}$$

From (7.2)-(9.2), it follows that

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} aX_0 + I & -aI \\ -X_0 & I \end{bmatrix} \begin{bmatrix} \exp(-aX_0)C_0 \\ \exp(-aX_0)C_1 \end{bmatrix}$$

From here it is clear that U satisfies the same initial conditions as X. From the uniqueness property for the solutions of the Cauchy problem (5.2), [11], it follows that $X(t) = U(t)$, and (2.2) is verified.

(ii) Let X be a solution of the equation (2.1) such that $C_0 = X(a)$ and $C_1 = X^{(1)}(a)$. Considering the operator function $U(t) = \exp(tX_0)T_1 + \exp(tX_1)T_2$, where T_1 and T_2 are unknown operators in $L(H)$, the Cauchy problem (5.2) is satisfied if T_1 and T_2 verify the conditions

$$\exp(aX_0)T_1 + \exp(aX_1)T_2 = C_0 \tag{10.2}$$

$$\exp(aX_0)X_0T_1 + \exp(aX_1)X_1T_2 = C_1$$

$$\begin{bmatrix} I & I \\ X_0 & X_1 \end{bmatrix} \begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_1) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \tag{11.2}$$

From the invertibility hypothesis of $X_1 - X_0$, and from the lemma 1 of [13], the operator matrix $S = \begin{bmatrix} I & I \\ X_0 & X_1 \end{bmatrix}$ is invertible and $S^{-1} = \begin{bmatrix} I + (X_1 - X_0)^{-1} & -(X_1 - X_0)^{-1} \\ -(X_1 - X_0)^{-1} X_0 & (X_1 - X_0)^{-1} \end{bmatrix}$. From (11.2) it follows that T_1 and T_2 are given by (4.2). From the uniqueness property for the solutions of the Cauchy problem (5.2), [11], the result is concluded.

The following result is concerned with the Sturm-Liouville problem (1.1). In accordance with the notation used in th. 2.1, we represent by S the operator matrix

$$S = \begin{bmatrix} I & I \\ X_0 & X_1 \end{bmatrix} \tag{12.2}$$

and if $X_1 - X_0$ is an invertible operator in $L(H)$, then S is invertible and

$$S^{-1} = \begin{bmatrix} I + (X_1 - X_0)^{-1} & -(X_1 - X_0)^{-1} \\ -(X_1 - X_0)^{-1} X_0 & (X_1 - X_0)^{-1} \end{bmatrix} \tag{13.2}$$

Theorem 2.2. Let us consider the problem (1.1), and let λ be a complex number such that the equation (3.1) has a pair of solutions X_0 and X_1, such that $X_1 - X_0$ is invertible. Let us denote by T the operator matrix

(14.2)

$$T = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} S \begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_1) \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} S \begin{bmatrix} \exp(X_0 b) & 0 \\ 0 & \exp(X_1 b) \end{bmatrix}$$

then

(i) If T is invertible in $L(H \oplus H)$, the only solution of the problem (1.1) is the trivial one, $X(t) = 0$, for all $t \in [a, b]$.

(ii) If $0 \in \sigma_{\text{comp}}(T)$, then there are nontrivial solutions of (1.1). These solutions take the form

$$X(t) = \exp(X_0 t)C + \exp(tX_1)D, \tag{15.2}$$

where C and D are operators in $L(H)$ satisfying

$$T \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16.2}$$

and T is given by (14.2). Moreover, if N is a closed subspace of $H \oplus H$ that is orthogonal to the subspace $T(H \oplus H)$, and N_1, N_2 are the subspaces of $H \oplus H$, defined by

$$N_1 = N \cap (H \oplus \{0\}), \quad N_2 = N \cap (\{0\} \oplus H)$$

then C and D can be chosen as the projections on H with ranges N_1 and N_2 respectively.

(iii) If H is finite-dimensional, there are nontrivial solutions of (1.1), if and only if, T is singular. In this case nontrivial solutions of (1.1) are obtained from (15.2) solving the algebraic system (16.2).

Proof. From th. 2.1-(ii), the general solution of the operator differential equation (2.1) takes the form expressed by (15.2). If we impose that $X(t)$ given by (15.2) satisfies (1.1), it follows that the operators C and D must verify the conditions

$$(17.2) \qquad M_{11}(\exp(aX_0)C + \exp(aX_1)D) + N_{11}(\exp(bX_0)C + \exp(bX_1)D) +$$
$$+ M_{12}(\exp(aX_0)X_0C + \exp(aX_1)X_1D) + N_{12}(\exp(bX_0)X_0C + \exp(bX_1)X_1D) = 0$$
$$M_{21}(\exp(aX_0)C + \exp(aX_1)D) + N_{21}(\exp(bX_0)C + \exp(bX_1)D) +$$
$$+ M_{22}(\exp(aX_0)X_0C + \exp(aX_1)X_1D) + N_{22}(\exp(bX_0)X_0C + \exp(bX_1)X_1D) = 0$$

The system (17.2) may be written in the following form

$$(18.2) \qquad \left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \exp(aX_0) & \exp(aX_1) \\ \exp(aX_0)X_0 & \exp(aX_1)X_1 \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \exp(bX_0) & \exp(bX_1) \\ \exp(bX_0)X_0 & \exp(bX_1)X_1 \end{bmatrix} \right) \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Considering the operator matrix S given by (12.2), from the hypothesis, S is invertible and S^{-1} is given by (13.2). From here (18.2) may be expressed in the form

$$\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} S \begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_1) \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} S \begin{bmatrix} \exp(bX_0) & 0 \\ 0 & \exp(bX_1) \end{bmatrix}\right) \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19.2}$$

Under the invertibility hypothesis of T, it is clear that the only solution of (19.2) is $C = D = 0$, thus the only solution of (1.1) is $X(t) = 0$, for all $t \in [a, b]$. Thus (i) is proved.

(ii) If $0 \in \sigma_{\mathrm{comp}}(T)$, then the subspace $T(H \oplus H)$ is not dense in $H \oplus H$. From here and (19.2) the result is concluded.

(iii) It is a consequence of (i), (ii) and the fact that for the finite- dimensional case $\sigma(T) = \sigma_{\mathrm{comp}}(T)$.

Example 1. Let us consider the problem (1.1) where $A_0 = -3I$, $A_1 = -I$, then the corresponding characteristic equation (3.1) takes the form

$$X^2 - X - 2I = 0$$

and $X_0 = 2I$, $X_1 = -I$, are two solutions of this equation satisfying $X_1 - X_0$ invertible. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis of H and the coefficient operators M_{ij} and N_{ij} for $1 \leq i, j \leq 2$, arising in (1.1) satisfy the property that their ranges are contained in the subspace $\mathrm{LIN}(\{e_n\}_{n\geq 1})$, then from (14.2) and (19.2), it is clear that if we take $C = D = P$, where P is the orthogonal projection on the subspace $\mathrm{LIN}(\{e_o\})$, the expression (15.2) with $C = D = P$, and $X_0 = 2I$, $X_1 = -I$, provides a nontrivial solution of the problem (1.1).

The following result deals with the study of the problem (1.1) when the algebraic equation (3.1) has a double root.

Theorem 2.3. Let us consider the problem (1.1) where λ is a complex paramenter and X_0 is a double root of (3.1), and let W be the operator matrix

$$W = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \exp(aX_0) & a\exp(aX_0) \\ \exp(aX_0)X_0 & (aX_0 + I)\exp(aX_0) \end{bmatrix} \tag{20.2}$$

$$+ \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \exp(bX_0) & b\exp(bX_0) \\ \exp(bX_0)X_0 & (bX_0 + I)\exp(bX_0) \end{bmatrix}$$

then the following results are verified.

(i) If W is invertible in $L(H \oplus H)$, the only solution of the problem (1.1) is the trivial one.

(ii) If $0 \in \sigma_{comp}(W)$, then there are nontrivial solutions of (1.1). These solutions may be expressed in the form

$$X(t) = \exp(tX_0)(C + tD) \tag{21.2}$$

where C and D are operators in $L(H)$ satisfying

$$W = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22.2}$$

If R is a closed subspace of $H \oplus H$ that is orthogonal to the subspace $W(H \oplus H)$, and R_1 and R_2 are the subspaces of $H \oplus H$, defined by

$$R_1 = R \cap (H \oplus \{0\}), \quad R_2 = N \cap (\{0\} \oplus H) \tag{23.2}$$

then C and D can be chosen as the projections on H with ranges R_1 and R_2 respectively.

(iii) If H is finite-dimensional, there are nontrivial solutions of (1.1), if and only if, W is singular. In this case, solving (22.2) and placing the solutions C, D, in the expression (21.2) one gets nontrivial solutions of (1.1).

Proof. From th.2.1-(i), the general solution of the equation (2.1) may be expressed in the form (21.2). If we impose that $X(t)$ given by (21.2) satisfies the boundary value conditions arising in (1.1), it follows that the operators C and D must verify the conditions

(24.2)

$$M_{11}(\exp(aX_0)(C+aD)+N_{11}(\exp(bX_0)(C+bD))+M_{12}(\exp(aX_0)(D+X_0C+aX_0D))$$
$$+N_{12}(\exp(bX_0)(D+X_0C+bX_0D))=0$$
$$M_{21}(\exp(aX_0)(C+aD)+N_{21}(\exp(bX_0)(C+bD))+M_{22}(\exp(aX_0)(D+X_0C+aX_0D))$$
$$+N_{22}(\exp(bX_0)(D+X_0C+bX_0D))=0$$

This system may be written in the form

(25.2)

$$(M_{11}\exp(aX_0)+M_{12}\exp(aX_0)X_0)C+(M_{11}a\exp(aX_0)+M_{12}(a\exp X_0+I)\exp(aX_0))D+$$
$$+(N_{11}\exp(bX_0)+N_{12}\exp(bX_0)X_0)C+(N_{11}b\exp(bX_0)+N_{12}(b\exp X_0+I)\exp(bX_0))D=0$$
$$(M_{21}\exp(aX_0)+M_{22}\exp(aX_0)X_0)C+(M_{21}a\exp(aX_0)+M_{22}(\exp(aX_0)(aX_0+I))D+$$
$$+(N_{21}\exp(bX_0)+N_{22}\exp(bX_0)X_0)C+(N_{21}b\exp(bX_0)+N_{22}(b\exp X_0+I)\exp(bX_0))D=0$$

or equivalently

$$\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \exp(aX_0) & a\exp(aX_0) \\ \exp(aX_0)X_0 & (aX_0+I)\exp(aX_0) \end{bmatrix}\right. \tag{26.2}$$
$$\left.+\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} \exp(bX_0) & b\exp(bX_0) \\ \exp(bX_0)X_0 & (bX_0+I)\exp(bX_0) \end{bmatrix}\right)\begin{bmatrix} C \\ D \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This proves (i).

(ii) If $0\in\sigma_{\text{comp}}(W)$, then the subspace $W(H\oplus H)$ is not dense in $H\oplus H$ and the results of (ii) are proved.

(iii) It is a consequence of (i), (ii) and the fact that for the finite- dimensional case one has $\sigma_{\text{comp}}(W)=\sigma(W)$.

Example 2. Note that if X_0 is a double root of the equation (3.1),that is such that $2X_0 + A_1 = 0$, then $X_0 = -A_1/2$. As X_0 has to be a solution of (3.1), the following condition has to be satisfied

$$0 = X_0^2 + A_1X_0 + A_0 - \lambda I = A_1^2/4 - A_1^2/2 + A_0 - \lambda I, \quad -A_1^2/4 + A_0 - \lambda I = 0 \tag{27.2}$$

Let us consider the problem (1.1) where A_0 and A_1 satisfy (27.2), then $X_0 = -A_1/2$ is a double root of (3.1). Let us suppose that $M_{12} = M_{21} = N_{21} = N_{12} = 0$, and $M_{11} = M_{22} = N_{11} = N_{22} = I$. Then the operator matrix W given by (20.2) takes the form

(28.2)
$$W = \begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_0) \end{bmatrix} \begin{bmatrix} I & aI \\ X_0 & aX_0 + I \end{bmatrix} + \begin{bmatrix} \exp(bX_0) & 0 \\ 0 & \exp(bX_0) \end{bmatrix} \begin{bmatrix} I & bI \\ X_0 & bX_0 + I \end{bmatrix}$$

From the lemma 1 of [13], it follows that

$$\begin{bmatrix} I & aI \\ X_0 & aX_0 + I \end{bmatrix}^{-1} = \begin{bmatrix} aX_0 + I & -aI \\ -X_0 & I \end{bmatrix}; \begin{bmatrix} I & bI \\ X_0 & bX_0 + I \end{bmatrix}^{-1} = \begin{bmatrix} bX_0 + I & -bI \\ -X_0 & I \end{bmatrix} \tag{29.2}$$

From (28.2) and (29.2), postmultiplying the right hand side of (28.2) by

$$\begin{bmatrix} bX_0 + I & -bI \\ -X_0 & I \end{bmatrix} \begin{bmatrix} \exp(-bX_0) & 0 \\ 0 & \exp(-bX_0) \end{bmatrix}$$

and taking into account the commutativity between $\begin{bmatrix} \exp(aX_0) & 0 \\ 0 & \exp(aX_0) \end{bmatrix}$ and $\begin{bmatrix} I & aI \\ X_0 & aX_0 + I \end{bmatrix}$, it follows that W is invertible in $L(H \oplus H)$, if and only if, the operator matrix

$$I + \begin{bmatrix} \exp((a-b)X_0) & 0 \\ 0 & \exp((a-b)X_0) \end{bmatrix} \begin{bmatrix} I + (b-a)X_0 & (a-b)I \\ (b-a)X_0^2 & I + (a-b)X_0 \end{bmatrix} \tag{30.2}$$

Is invertible in $L(H \oplus H)$. As $X_0 = -A_1/2$, from (30.2) and th. 2.3, it follows that for the finite-dimensional case, there are nontrivial solutions of (1.1), if and only if the following operator matrix is singular

$$I+\begin{bmatrix}\exp(A_1(b-a)/2) & 0\\ 0 & \exp(A_1(b-a)/2)\end{bmatrix}\begin{bmatrix}I-A_1(b-a)/2 & (a-b)I\\ A_1^2(b-a)/4 & I+(a-b)A_1/2\end{bmatrix}$$

$$=\begin{bmatrix}I+(I-(b-a)A_1/2)\exp(A_1(b-a)/2) & (a-b)\exp(A_1(b-a)/2)\\ ((b-a)A_1^2/4)\exp(A_1(b-a)/2) & I+(I-(a-b)A_1/2)\exp(A_1(b-a)/2)\end{bmatrix}$$

Under this condition, solving the system (22.2) with W given by (20.2), and placing the solutions C, D into the expression (21.2), nontrivial solutions of (1.1) are obtained.

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