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FUZZY RELATION EQUATIONS UNDER LSC AND USC T-NORMS AND THEIR BOOLEAN SOLUTIONS

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SUMMARY

This paper follows a companion paper (Stochastica, 8 (1984), 99-145) in which we gave the state of the art of the theory of fuzzy relation equations under a special class of triangular norms. Here we continue this theory establishing new results under lower and upper semicontinuous triangular norms and surveying on the main theoretical results already appeared in foregoing papers. Max-t fuzzy equations with Boolean solutions are recalled and studied. Many examples clarify the results established.

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1. Introduction.

In [10], we gave the main theoretical and applicational aspects of fuzzy relation equations theory, showing that substituting the classical operator "min" of the unit real interval with a suitable triangular norm t of Schweizer and Sklar [27], one obtains an extended form of the usual max-min fuzzy equations introduced by Sanchez [25].

In the Sections 6 and 7 of [10], we characterized the greatest solution and the minimal solutions of a max-t fuzzy equation. In the Sections 6 and 7 of [10], we studied the solutions with the smallest fuzziness measure of Yager [34] valued by means of a triangular norm and conorm. In the Sections 8 and 9 of [10], we illustrated some applications to the analysis of the fuzzy systems and of some decision-making processes in fuzzy environment.

Since many results appeared in the literature, it have seemed again quite natural, with the same spirit with which we wrote the first survey [10], to collect these results in this second work in order to give the widest possible point of view on this theory.

For commodity of the reader, we exhibit, although with slight modifications, the proofs of the theorems already established in previous papers. The results of [10] are only recalled and new results are also given.

Section 2 contains some improvements of Section 2 of [10] in connection with the results of Gottwald [15], [16]. Section 3 contains the basic preliminaries. Results of max-t fuzzy equations under lower and upper semicontinuous triangular norms are in Section 4 and 5, respectively. Section 6 contains some general algebraic results and a condition which guarantees the existence of Boolean solutions of a fuzzy equation under any triangular norm t. In particular, if t is lower semicontinuous, we determine the greatest Boolean solution in Section 7. In the Sections 8 and 9, we study minimal Boolean solutions for max-t fuzzy equations under triangular norms that are continuous in both variables.

For uniformity of presentation, we adopt the same symbology used in [10].

2. Norms, ψ-operators and β-operators.

A triangular norm (briefly, norm) t [27] is a real function t: $[0,1] \times [0,1] \to [0,1]$ of two variables with the properties $at0 = 0$, $at1 = a$, $at(btc) = (atb)tc$, $atb \leq a'tb'$ if $a \leq a'$ and $b \leq b'$, where $atb = t(a,b)$, a, a', b, b', $c \in [0,1]$.

By putting

$$\mathbf{I}_t(a,b) = \{x \in [0,1] : atx \leq b\}$$

for any $a, b \in [0,1]$, following [20], we defined in [10] the operator $\psi_t : [0,1] \times [0,1] \to [0,1]$, connected with the norm t, as

$$a\psi_t b = \sup\{x \in I_t(a,b)\} \tag{2.1}$$

for any $a, b \in [0,1]$, where $a\psi_t b = \psi_t(a,b)$.

It is easily seen that [15], [20], [24],

$$a\psi_t b \leq a\psi_t \quad \text{if } b \leq c, \tag{2.2}$$

$$a\psi_t(atb) \geq b, \tag{2.3}$$

for any $a, b \in [0,1]$. Note that $\mathbf{I}_t(a,b) \neq \emptyset$ for any $a, b \in [0,1]$ since $0 \in \mathbf{I}_t(a,b)$. In general, the position (2.1) does not assure that $a\psi_t b$ belongs to the set $\mathbf{I}_t(a,b)$ as it was shown in the Example 1 of [10]. It is obvious that the belongness of $a\psi_t b$ to the set $\mathbf{I}_t(a,b)$ is guaranteed if one assumes

$$at(a\psi_t b) \leq b \tag{2.4}$$

for any $a, b \in [0,1]$. In the sequel, when no misunderstanding can arise, we put $a\psi b$ instead of $a\psi_t b$.

We now restrict our attention to the norms which are lower semicontinuous (briefly, LSC). As pointed out in [15], [16], since the norms are monotone and symmetric, the lower semicontinuity is equivalent to the left continuity in both arguments. Further if T is LSC, the following equality

$$xt(\sup y_i) = \sup(xty_i) \tag{2.5}$$

holds for any family $\{y_i\}$ of real numbers of [0,1] and for any $x \in [0,1]$. We now show that

Theorem 1. If t is LSC, then (2.4) holds. Further, (2.4) and (2.5) are equivalent.

Proof. If t is LSC, then (2.5) holds. Thus

$$at(a\psi b) = at(\sup\{z \in [0,1] : atz \leq b\}) = \sup\{atz : atz \leq b\} \leq b$$

for any $a, b \in [0,1]$ and hence (2.4) is satisfied.

Assume now that (2.4) holds for any $a, b \in [0,1]$ and let $\{y_i\}$ be a family of real numbers of [0,1]. Since $y_i \leq \sup y_i$ for any index i, we have by the monotonicity of t,

$$\sup(xty_i) \leq xt(\sup y_i) \tag{2.6}$$

for any $x \in [0,1]$. Thus, from

$$xty_i \leq \sup(xty_i),$$

it follows that

$$\sup y_i \leq x\psi(\sup(xty_i))$$

and hence by (2.4),

$$xt(\sup y_i) \leq xt(x\psi(\sup(xty_i)) \leq \sup(xty_i)$$

for any $x \in [0,1]$. This inequality and (2.6) prove that (2.5) holds.

It is clear that (2.1) can be used in order to define the operator ψ_t connected with a norm t, even if t is not necessarily LSC. But, since the property (2.4) plays (and played in [10]) an important key role in the whole theory of fuzzy relation equations, we shall assume explicitly in some Sections of this paper the lower semicontinuity of the norms under discussion.

Here we signal the works [2], [18], where the authors study sufficient conditions for the determination of ψ-operators and see also Weber [31], Dubois and Prade [13] and Mizumoto [22] for further studies and applications of the triangular norms.

In [10], according to [20], we assumed that

$$\text{for any } a \in [0,1],\ t(a, \cdot) \text{ is continuous in } [0,1]. \tag{2.7}$$

Of course, by symmetry, t is continuous also in the first variable and in accordance to the above establishments, the lower semicontinuity of t (or equivalently the left continuity in both arguments) is a more general assumption than (2.7). We also recall that Miyakoshi and Shimbo [20] have established a one-to-one correspondence between the set of all the left-hand continuous norms and the set of all right-hand continuous operators ψ satisfying suitable properties. Further, we point out that if t is LSC, then

$$\mathbf{I}_t(a,b) = [0, a\psi b]$$

and

$$a\psi b = 1 \quad \text{iff } a \le b.$$

Remark 1. It is evident that, following the terminology of lattice theory from the Birkhoff's book [1], the structure $([0,1], \bigwedge, \bigvee, \le, t, 0, 1)$, if t is LSC, is a bounded residuated lattice, with respect to which "t" is a binary isotone and commutative multiplication and ψ_t is the related operation of residuation. For other algebraic aspects, see [16].

In the sequel, we shall use in some numerical examples the following norms and respective ψ-operators:

$$at_1b = a \bigwedge b = \min(a,b), \quad a\psi_1 b = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{if } a > b, \end{cases}$$

$$at_2b = a \cdot b, \quad a\psi_2 b = \begin{cases} 1 & \text{if } a \le b, \\ b/a & \text{if } a > b, \end{cases}$$

$$at^{(p)}b = 1 - \min\{1, [(1-a)^p + (1-b)^p]^{1/p}\}, p \ge 1,$$

$$a\psi^{(p)}b = \begin{cases} 1 & \text{if } a \le b, \\ 1 - [(1-b)^p - (1-a)^p]^{1/p} & \text{if } a > b, \end{cases}$$

$$at_4b = \max\{0, a+b-1\}, \; a\psi_4 b = \begin{cases} 1 & \text{if } a \le b, \\ 1+b-a & \text{if } a > b, \end{cases}$$

$$at_5b = \begin{cases} 0 & \text{if } a+b-1 \le 0, \\ a \bigwedge b & \text{if } a+b-1 > 0, \end{cases} \quad a\psi_5 b = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{if } 1-a \le b < a, \\ 1-a & \text{if } b < \min\{a, 1-a\}, \end{cases}$$

where $a, b \in [0,1]$.

For t_1, t_2 see Zadeh [35], for t_3 see Yager [33], for t_4 see Giles [14], for t_5 see [2]. We note that ψ_1 is the well known Gödelian implication operator used by Sanchez [25] and $t_1, t_2, t^{(p)}, t_4$ are continuous while t_5 is LSC.

In [10], we define the set

$$\mathbf{G}_t(a,b) = \{x \in [0,1] : atx = b\}$$

for any $a, b \in [0,1]$. If $\mathbf{G}_t(a,b) \neq \emptyset$, where t is any norm, then we have $a = at1 \geq atx = b$ for any $x \in \mathbf{G}_t(a,b)$ and hence $a \geq b$. If $a \geq b$, and t is a norm satisfying property (2.7), it is easily seen that (cfr. Lemma 2.1 of [10]) $\mathbf{G}_t(a,b) \neq \emptyset$. If t is LSC and $a \geq b$, the set $\mathbf{G}_t(a,b)$ could be empty as it is proved in the following example.

Example 1. Let $t = t_5$, $a = 0.3 > 0.2 = b$. For $x \geq 0.7$, we have $a + x - 1 \leq 0$ and hence $at_5x = 0 \neq 0.2 = b$. For $x > 0.7$, we have $a + x - 1 > 0$ and hence $at_5x = a \bigwedge x = a = 0.3 \neq 0.2 = b$. This implies that $\mathbf{G}_t(0.3, 0.2) \neq \emptyset$.

However, assuming that t is LSC and $\mathbf{G}_t(a,b) \neq \emptyset$ for some $a, b \in [0,1]$ $(a \geq b)$, we have of course $\mathbf{G}_t(a,b) \subseteq \mathbf{I}_t(a,b)$. Let $x \in \mathbf{G}_t(a,b)$ and since t is nondecreasing, we deduce that $b = atx \leq at(a\psi b) \leq b$ by (2.4). This means that $a\psi b \in \mathbf{G}_t(a,b)$ (cfr. Lemma 2.2 of [10]) and then

$$a\psi b = \sup\{x \in \mathbf{G}_t(a,b)\}.$$

We now consider an upper semicontinuous (briefly, USC) norm t. By putting

$$\mathbf{H}_t(a,b) = \{x \in [0,1] : atx \geq b\}$$

for any $a, b \in [0,1]$, we note that $1 \in \mathbf{H}_t(a,b)$ if $a \geq b$ and $\mathbf{H}_t(a,b) = \emptyset$ if $a < b$ since $a = at1 \geq atx \geq b$ for any $x \in \mathbf{H}_t(a,b)$. We define the following operator $\beta_t : [0,1] \times [0,1] \rightarrow [0,1]$, dual of (2.1), as

$$a\beta_t b = \begin{cases} 0 & \text{if } a < b, \\ \inf\{x \in \mathbf{H}_t(a,b)\} & \text{if } a \geq b, \end{cases} \tag{2.8}$$

for any $a, b \in [0,1]$. From now on, when no misunderstanding can arise, we put $a\beta b$ instead of $a\beta_t b$. It is immediately seen that $a\beta b \leq a\beta c$ if either $b \leq c \leq a$ or a $\leq b \leq c$, $a\beta(atb) \leq b, a = 1\beta a$ and $0 = a\beta 0$, where $a, b, c \in [0,1]$.

Further, if t is USC and $a \geq b$, we have that

$$at(a\beta b) \geq b \text{ if } a \geq b. \tag{2.9}$$

Of course, $\mathbf{H}_t(a,b) = [a\beta b, 1]$ if t is USC and (2.8) can be also used for norms that are not necessarily USC. We note that if t is not USC, then $a\beta b$ generally, if $a \geq b$, does not belong to set $\mathbf{H}_t(a,b)$, as it is proved in easy examples. For instance, it suffices to consider $t = t_5$ (which is not USC because discontinuous [2]) and $a = b = 0.5$.

In the sequel, we shall consider USC norms and in some examples, the well known "drastic product" defined

$$at_6 b = \begin{cases} a & \text{if } b = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and whose related b_6-operator is given by

$$a\beta_6 b = \begin{cases} 0 & \text{if } a < b \text{ or } b = 0, \\ b & \text{if } 1 = a > b, \\ 1 & \text{if } 1 > a > b \text{ or } a = b, \end{cases}$$

where $a, b \in [0,1]$. Note that t_6 is USC and since it is discontinuous, it is not LSC.

Further, we point out that, for the norm t_1, β_1 coincides with the operator σ introduced by Sanchez [26].

We note that if t is USC and $a \geq b$, $\mathbf{G}_t(a,b)$ could be empty as it is proved in the following example.

Example 2. Let $t = t_6$ and $1 > a > b > 0$. We have $atx = 0$ if $x < 1$ and $atx = a > b$ if $x = 1$. Hence $\mathbf{G}_t(a,b) = \emptyset$.

However, assuming that t is USC and $\mathbf{G}_t(a,b) \neq \emptyset$ for some $a, b \in [0,1]\,(a \geq b)$, we have obviously $\mathbf{G}_t(a,b) \subseteq \mathbf{H}_t(a,b)$. Let $x \in \mathbf{G}_t(a,b)$ and since t is nondecreasing, we get

$b = atx \geq at\,(a\beta b) \geq b$ by (2.9). This implies that $a\beta b \in \mathbf{G}_t(a, b)$ and then

$$a\beta b = \inf\{x \in \mathbf{G}_t(a, b)\}.$$

Of course, if t is a norm satisfying property (2.7), we have that

$$\mathbf{G}_t(a, b) = [a\beta b, a\psi b].$$

It was also stressed in [10] that if a norm t, satisfying (2.7), is strictly increasing on the support set $\{x \in [0, 1] : atx > 0\}$ for any $a > 0$, then $\mathbf{G}_t(a, b) = \{a\beta b\} = \{a\psi b\}$ for $a > b > 0$.

Since the above norms t_1, t_2, $t^{(p)}$, t_4 are continuous, they satisfy this condition. We like to point out that some partial results of this Section, as well as Lemmas 2.1, 2.2, Examples 1, 2 and the results of p. 107 of [10] are due to Miyakoshi and Shimbo, appeared in their preprint, prior of the publication of the actual paper [20].

3. Basic definitions.

From now on, $\mathbf{I}_r$ denotes the set of the first r positive integers. Let $\mathbf{X} = \{x_1, x_2, \cdots, x_n\}$, $\mathbf{Y} = \{y_1, y_2, \cdots, y_m\}$, $\mathbf{Z} = \{z_1, z_2, \cdots, z_p\}$ be three finite sets and $\mathbf{F}(\mathbf{X}) = \{A : \mathbf{X} \in [0, 1]\}$, $\mathbf{F}(\mathbf{Y}) = \{B : \mathbf{Y} \rightarrow [0, 1]\}$ the sets of all the fuzzy sets of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Further, let $\mathbf{F}(\mathbf{X} \times \mathbf{Y}) = \{Q : \mathbf{X} \times \mathbf{Y} \rightarrow [0, 1]\}$, $\mathbf{F}(\mathbf{Y} \times \mathbf{Z}) = \{R : \mathbf{Y} \times \mathbf{Z} \rightarrow [0, 1]\}$ and $\mathbf{F}(\mathbf{X} \times \mathbf{Z}) = \{T : \mathbf{X} \times \mathbf{Z} \rightarrow [0, 1]\}$ be the sets of all the fuzzy relations defined in the specified domains.

It is well known that $\mathbf{F}(\mathbf{X})$ is a complete lattice with respect to the following pointwise operations:

$$(A \bigvee A')(x) = \max\{A(x), A'(x)\}, \quad \text{(fuzzy union)}$$

$$(A \bigwedge A')(x) = \min\{A(x), A'(x)\}, \quad \text{(fuzzy intersection)}$$

for any $x \in \mathbf{X}$, where $A, A' \in \mathbf{F}(\mathbf{X})$. Analogous operations shall be considered between fuzzy relations. For brevity of notations, we put for any $A \in \mathbf{F}(\mathbf{X})$, $B \in \mathbf{F}(\mathbf{Y})$, $Q \in \mathbf{F}(\mathbf{X} \in \mathbf{Y})$, $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$, $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$:

$$A(x_i) = A_i, B(y_j) = B_j, Q(x_i, y_j) = Q_{ij}, R(y_j, z_k) = R_{jk}, T(x_i, z_k) = T_{ik},$$

where $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$. We represent fuzzy sets and fuzzy relations as real matrices and we use the following definitions of [10].

Definition 3.1. Let $M, M' \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ (resp. $A, A' \in \mathbf{F}(\mathbf{X})$). We say that M(resp. A) is contained in M' (resp. A'), in symbols $M \leq M'$ (resp. $A \leq A'$) if $M_{ij} \leq M_{ij}$ (resp. $A_i \leq A_i$) for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$ (resp. $i \in \mathbf{I}_n$).

Definition 3.2. We define inverse of $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$, the fuzzy relation $Q^{-1} \in \mathbf{F}(\mathbf{Y} \times \mathbf{X})$ with membership function $Q^{-1}_{ji} = Q_{ij}$ for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$.

Definition 3.3. Let $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$, $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$ and t be a norm. We define sup $-t$ composition of R and Q, the fuzzy relation $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$, $T = R \perp_t Q$, given by

$$T_{ik} = \bigvee_{j=1}^{m} [Q_{ij} t R_{jk}] \tag{3.1}$$

for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$.

Definition 3.4. The sup $-t$ composition of $A \in \mathbf{F}(\mathbf{X})$ and $M \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ is the fuzzy set $B \in \mathbf{F}(\mathbf{Y})$, $B = M \perp_t A$, given by

$$B_j = \bigvee_{i=1}^{n} [A_i t M_{ij}] \tag{3.2}$$

for any $j \in \mathbf{I}_m$.

Definition 3.5. Let t be a LSC norm. The Ψ-composition of $Q^{-1} \in \mathbf{F}(\mathbf{Y} \times \mathbf{X})$ and $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$ is the fuzzy relation $Q^{-1} \Psi T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$ given by

$$(Q^{-1} \Psi T)_{jk} = \bigwedge_{i=1}^{n} (Q^{-1}_{ji} \Psi T_{ik})$$

for any $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$.

Definition 3.6. Let t be a LSC norm. The Ψ-composition of $A \in \mathbf{F}(\mathbf{X})$ and $B \in \mathbf{F}(\mathbf{Y})$ is the fuzzy relation $A\Psi B \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ given by

$$(A\Psi B)_{ij} = A_i \Psi B_j$$

for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$.

Definition 3.7. We define lower (resp. upper) solution of a fuzzy equation, a minimal (resp. maximal) element of the set, ordered by fuzzy inclusion in accordance to Def. 3.1, of its solutions.

If $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ and $T \in \mathbf{F}(\mathbf{X} \in \mathbf{Z})$ (resp. $A \in \mathbf{F}(\mathbf{Y})$) are assigned, we denote by $\mathcal{R} = \mathcal{R}(Q, T)$ (resp. $\mathcal{M} = \mathcal{M}(A, B)$) the set of all the fuzzy relations $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$ (resp. $M \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$) satisfying the Equation (3.1) (resp. (3.2)).

As in [10], we put

$$Q_i(x_i, y_i) = Q_{ij} \text{ and } T_i(x_i, z_k) = T_{ik}$$

for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$ and it was pointed out that, assuming $Q_i \in \mathbf{F}(\{x_i\} \times \mathbf{Y})$ and $T_i \in \mathbf{F}(\{x_i\} \times \mathbf{Z})$ as fuzzy sets for any $i \in \mathbf{I}_n$, the fuzzy equation (3.1) can be seen as a system of n equations of type (3.2),

$$T_h = R \bot_t Q_h, \tag{3.3}$$

where $h \in \mathbf{I}_n$. Then, if $\mathcal{M}_h = \mathcal{M}(Q_h, T_h)$ denotes the set of all the solutions $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$ of the Equation (3.3) for any $h \in \mathbf{I}_n$, we have

$$\mathcal{R} = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_n.$$

In the sequel, we need the following well known Lemma 3.7 of [10]:

Lemma 3.1. Let R_1, $R_2 \in \mathcal{R}$ (resp. $\mathcal{M}$) and $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$ such that $R_1 \leq R \leq R_2$. Then $R \in \mathcal{R}$ (resp. $\mathcal{M}$).

4. Fuzzy equations under LSC norms.

In [10], as well as in [20], [24], the following results were proved for the Equations (3.1) and (3.2) under norms satisfying property (2.7).

Theorem 4.1. $\mathcal{R} \neq \emptyset$ iff $Q^{-1}\Psi T \in \mathcal{R}$. Further, $Q^{-1}\Psi T \geq R$ for any $R \in \mathcal{R}$.

Theorem 4.2. $\mathcal{M} \neq \emptyset$ iff $A\Psi B \in M$. Further, $A\Psi B \geq M$ for any $M \in \mathcal{M}$.

Theorem 4.1. of which Theorem 4.2 is a particular case, was proved in [10] using essentially the properties (2.2), (2.3) and (2.4).

Hence it is evident that it holds for max-t fuzzy equations under LSC norms since property (2.4) holds by Theorem 2.1.

Example 3. Let $t = t_5$, $m = n = 2$, $A \in \mathbf{F}(\mathbf{X})$ and $B \in \mathbf{F}(\mathbf{Y})$ be given by

$$A = \begin{matrix} y_1 & y_2 \\ \end{matrix} \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \quad B = \begin{matrix} y_1 & y_2 \\ \end{matrix} \begin{pmatrix} 0.5 & 0.0 \end{pmatrix}$$

Thus $\mathcal{M} \neq \emptyset$ since $(A\Psi_5 B)\perp_{t_5} A = B$ where

$$A\Psi_5 B = \begin{matrix} & y_1 & y_2 \\ x_1 & 1.0 & 0.5 \\ x_2 & 1.0 & 0.5 \end{matrix}$$

Note that the equation

$$B = M\perp_{t_5} A$$

has no lower solutions. Indeed, we have the following system of equations for any $M \in \mathcal{M}$:

$$\begin{cases} (0.5t_5 M_{11}) \bigvee (0.5t_5 M_{21}) = 0.5 \\ (0.5t_5 M_{12}) \bigvee (0.5t_5 M_{22}) = 0 \end{cases}$$

that is satisfied iff

$$\text{either} \begin{cases} 0.5t_5 M_{11} = 0.5, \\ 0.5t_5 M_{21} \leq 0.5, \\ 0.5t_5 M_{12} = 0, \\ 0.5t_5 M_{22} = 0, \end{cases} \quad \text{or} \begin{cases} 0.5t_5 M_{11} \leq 0.5, \\ 0.5t_5 M_{21} = 0.5, \\ 0.5t_5 M_{12} = 0, \\ 0.5t_5 M_{22} = 0, \end{cases}$$

i.e. iff either $M_{11} \in (0.5, 1]$, $M_{21} \in [0, 1]$ or $M_{11} \in [0, 1]$, $M_{21} \in (0.5, 1]$ and $M_{12}, M_{22} \in [0, 0.5]$. For $x \in [0.5, 1]$, we put

$$M_x = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} x & 0.0 \\ 0.0 & 0.0 \end{pmatrix} \end{array} \quad M_x = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 \\ x & 0.0 \end{pmatrix} \end{array}$$

Thus M_x, $M_x \in \mathcal{M}$ if $x > 0.5$ but $M_{0.5}$ and $M_{0.5}$ do not belong to $\mathcal{M}$.

Summarizing, we can say that if t is LSC, the Equations (3.1) and (3.2) have unique upper solution but in general, as Example 3 shows, do not have lower solutions. However, the membership functions of A and B can be such that some lower solution may exist as it is easily seen in the following example.

Example 4. Let $t = t_5$, $m = n = 2$, $A \in \mathbf{F(X)}$ be given by $A_1 = 0.6$, $A_2 = 0.5$ and B as in Example 3. Since $(A\Psi_5 B)\bot_{t_5} A = B$, where

$$A\Psi_5 B = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.5 & 0.4 \\ 1.0 & 0.5 \end{pmatrix} \end{array}$$

then $\mathcal{M} \neq \emptyset$. Reasoning as in Example 3, one can see that the fuzzy relations

$$N = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.0 \end{pmatrix} \end{array} \quad M_x = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \end{array}$$

where $x \in [0.5, 1]$, are elements of $\mathcal{M}$ if $x > 0.5$. N is minimal in $\mathcal{M}$ but $M_{0.5} \notin \mathcal{M}$.

Remark 2. In account of Remark 1, it is seen that Theorem 4.1 and 4.2 can be deduced by Theorem 1 of Di Nola and Lettieri [5], result valid also in the context of complete residuated lattices.

5. <u>Fuzzy relations under USC norms.</u>

Let t be an USC norm and, considering the Equation (3.2), we define the following m sets:

$$K_j = \{i \in \mathbf{I}_n : A_i t(A_i \beta B_j) = B_j\}$$

for any $j \in \mathbf{I}_m$. The following result holds.

Theorem 5.1. $\mathcal{M} \neq \emptyset$ iff $\mathbf{K}_j \neq \emptyset$ for any $j \in \mathbf{I}_m$.

Proof. Let $\mathcal{M} \neq \emptyset$. Then we have for any $M \in \mathcal{M}$ and $j \in \mathbf{I}_m$:

$$B_j = \bigvee_{i=1}^{n} [A_i t M_{ij}] = A_h t M_{hj}$$

for some $h \in \mathbf{I}_n$. Thus $M_{hj} \in \mathbf{G}_t(A_h, B_j) \neq \emptyset$ and hence, in accordance to the results of Section 2, we have that $A_h \beta B_j \in \mathbf{G}_t(A_h, B_j)$, i.e.

$$A_h t(A_h \beta B_j) = B_j$$

and this means that $\mathbf{K}_j \neq \emptyset$ since $h \in \mathbf{K}_j$. Vice versa, let $\mathbf{K}_j \neq \emptyset$ for any $j \in \mathbf{I}_m$. Define the following fuzzy relation $A\Theta B \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ with membership function given by (cfr. p. 117 of [10]):

$$(A\Theta B)_{ij} = \begin{cases} A_i \beta B_j & \text{if } i \in \mathbf{K}_j,\ B_j > 0, \\ 0 & \text{if } i \in \mathbf{K}_j,\ B_j > 0, \\ 0 & \text{if } B_j = 0. \end{cases} \tag{5.1}$$

for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$. We have for any $j \in \mathbf{I}_m$ such that $B_j > 0$:

$$\bigvee_{i=1}^{n} [A_i t(A\Theta B)_{ij}] = \{ \bigvee_{i \in \mathbf{K}_j} [A_i t(A_i \beta B_j)] \} \bigvee \{ \bigvee_{i \notin \mathbf{K}_j} (A_i t 0) \} = \{ \bigvee_{i \in \mathbf{K}_j} B_j \} \bigvee 0 = B_j.$$

If $B_j = 0$, the thesis is trivial.

Note that if t satisfies property (2.7), then $i \in \mathbf{K}_j$ iff $A_i \geq B_j$ and hence Theorem 5.1 becomes Theorem 4.1 of [10] for the necessary part and Theorem 4.3 of [10] for the sufficient part.

From the proof of Theorem 5.1, we can also say that

Theorem 5.2. $\mathcal{M} \neq \emptyset$ iff $A\Theta B \in \mathcal{M}$.

Proof. If $\mathcal{M} \neq \emptyset$, then $\mathbf{K}_j \neq \emptyset$ for any $j \in \mathbf{I}_m$ by Theorem 5.1. Consequently one proves that the fuzzy relation $A\Theta B$, defined in (5.1), belongs to $\mathcal{M}$.

Adopting the same proofs of Theorems 4.2, 4.4, 4.7 of [10] respectively, we can show the following results.

Theorem 5.3. If $\mathcal{M} \neq \emptyset$, then $\mathcal{M}$ has minimal elements L obtained by choosing an index $h \in \mathbf{K}_j$ for any $j \in \mathbf{I}_m$ and by putting for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$:

$$L_{ij} = \begin{cases} A_h \beta B_j & \text{if } B_j > 0,\, i = h, \\ 0 & \text{if } B_j > 0,\, i \neq h, \\ 0 & \text{if } B_j = 0. \end{cases}$$

In other words, it suffices, in order to determine a lower solution L, to keep a non-zero element $A_i \beta B_j$ in the $j-th$ column of $A\Theta B$ for which $B_j > 0$ and $i \in \mathbf{K}_j$. If $B_j = 0$, then we assume $L_{ij} = 0$ for any $i \in \mathbf{I}_n$.

Theorem 5.4. If $\mathcal{M} \neq \emptyset$, the fuzzy union of all the minimal elements of $\mathcal{M}$ is $A\Theta B$.

Theorem 5.5. If $\mathcal{M} \neq \emptyset$, for any $M \in \mathcal{M}$ there exists a minimal element $L \leq \mathcal{M}$ such that $L \leq M$.

Example 5. Let $t = t_6$, $m = n = 3$, $A \in \mathbf{F}(\mathbf{X})$ and $B \in \mathbf{F}(\mathbf{Y})$ be given by

$$A = \begin{pmatrix} \overset{x_1}{0.2} & \overset{x_2}{0.5} & \overset{x_3}{1} \end{pmatrix} \quad B = \begin{pmatrix} \overset{y_1}{0.8} & \overset{y_2}{0.5} & \overset{y_3}{0} \end{pmatrix}$$

We have

$$A\Theta_6 B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.8 & 0.5 & 0.0 \end{pmatrix} \end{array}$$

We get $\mathcal{M} \neq \emptyset$ since

$$(A\Theta_6 B) \bot_{t_6} A = B.$$

Note that $\mathbf{K}_1 = \{1\}$, $\mathbf{K}_2 = \{2,3\}$, $\mathbf{K}_3 = \mathbf{I}_3$ and then $\mathcal{M}$ has the following minimal elements:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.8 & 0.0 & 0.0 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.8 & 0.5 & 0.0 \end{pmatrix} \end{array}.$$

It is easily seen also here, as in example 3, that $\mathcal{M}$ has not the greatest element.

Summarizing, we can say that if t is USC, the set of all the lower solutions of the Equation (3.2) is completely determined, but in general, as Example 5 shows, this equation has not an upper solution. However, in certain particular cases, the membership functions of A and B can be such that an upper solution exists, as it is easily seen with simple examples.

About the Equation (3.1), we assume that $\mathcal{R} \neq \emptyset$. By (3.4), we have $\mathcal{M}_h \neq \emptyset$ for any $h \in \mathbf{I}_n$. Thus the set $\mathbf{L}_h = \mathbf{L}(Q_h, T_h)$ of all the lower solutions of each Equation (3.3) is nonempty by Theorem 5.3. If L_h denotes an arbitrary element of $\mathbf{L}_h$, we define the following subset of $\mathbf{F}(\mathbf{Y} \times \mathbf{Z})$:

$$\mathbf{L} = \mathbf{L}(Q,T) = \{L \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z}) : L = \bigvee_{h=1}^{n} L_h,\, L_h \in \mathbf{L}_h\}.$$

Then the following result holds.

Theorem 5.6. $\mathcal{R} \neq \emptyset$ iff $\mathcal{R} \cap \mathbf{L} \neq \emptyset$.

Proof. Let $R \in \mathcal{R}$. By (3.4), R belongs to $\mathcal{M}_h$ for any $h \in \mathbf{I}_n$. By Theorem 5.5, there exists an element $L_h \in \mathbf{L}_h$ such that $L_h \leq R$ for any $h \in \mathbf{I}_n$.

This implies that

$$L = \bigvee_{h=1}^{n} L_h \leq R,$$

where, of course, L belongs to $\mathbf{L}$. Since $L_h \leq L \leq R$ for any $h \in \mathbf{I}_n$, L belongs to $\mathcal{M}_h$ for any $h \in \mathbf{I}_n$ by Lemma 3.1 and this means that L is in $\mathcal{R}$ since (3.4) holds. Then $\mathcal{R} \cap \mathbf{L} \neq \emptyset$ since L is in $\mathcal{R} \cap \mathbf{L}$.

The converse implication is trivial.

In other words, this theorem assures that for any $R \in \mathcal{R}$, there exists an element $L \in \mathbf{L}$ such that $L \leq R$. Further, we have that

Theorem 5.7. R is minimal in $\mathcal{R}$ iff R is minimal in $\mathcal{R} \cap \mathbf{L}$.

Proof. It suffices only to show that if R is minimal in $\mathcal{R} \cap \mathcal{L}$, then R is minimal in $\mathcal{R}$. Indeed, let $R' \in \mathcal{R}$ such that $R' \leq R$. Then there exists an element $L \in \mathcal{R} \cap \mathbf{L}$ such that $L \leq R'$.

This implies that $L \leq R' \leq R$ and hence $L = R$ since R is minimal in $\mathcal{R} \cap \mathbf{L}$. Then $R = L = R'$ and thus R is minimal in $\mathcal{R}$ too.

Since each set $\mathbf{L}_h$, $h \in \mathbf{I}_n$, is finite, $\mathbf{L}$ is also a finite set and consequently the set $R \cap \mathbf{L}$ is finite. It has minimal elements, that are the minimal elements of $\mathcal{R}$ by Theorem 5.7, as it is shown in the following example.

Example 6. Let $t = t_6$, $n = p = 2$, $m = 3$, $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ and $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$ be given by

$$Q = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.9 & 1.0 & 0.6 \\ x_2 & 0.7 & 0.5 & 1.0 \end{matrix}, \quad T = \begin{matrix} & z_1 & z_2 \\ x_1 & 0.8 & 0.6 \\ x_2 & 0.2 & 0.7 \end{matrix},$$

We have

$$Q_1 \Theta_6 T_1 = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.8 & 0.6 \\ y_3 & 0.0 & 1.0 \end{matrix}, \quad Q_2 \Theta_6 T_2 = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 1.0 \\ y_2 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.7 \end{matrix}.$$

Then $\mathbf{L}_1 = \{L_1', L_1''\}$, $\mathbf{L}_2 = \{L_2', L_2''\}$, where

$$L_1' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.8 & 0.6 \\ y_3 & 0.0 & 0.0 \end{matrix}, \quad L_1'' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.8 & 0.0 \\ y_3 & 0.0 & 1.0 \end{matrix},$$

$$L_2' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 1.0 \\ y_2 & 0.0 & 0.0 \\ y_3 & 0.2 & 0.0 \end{matrix}, \quad L_2'' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.0 & 0.0 \\ y_3 & 0.2 & 0.7 \end{matrix}.$$

Then $\mathbf{L} = \{L, R, U, W\}$, where

$$L = L_1' \bigvee L_2'' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 1.0 \\ y_2 & 0.8 & 0.6 \\ y_3 & 0.2 & 0.0 \end{matrix}, \quad R = L_1' \bigvee L_2'' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.8 & 0.6 \\ y_3 & 0.2 & 0.7 \end{matrix},$$

$$U = L_1' \bigvee L_2' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 1.0 \\ y_2 & 0.8 & 0.0 \\ y_3 & 0.2 & 1.0 \end{matrix}, \quad W = L_1'' \bigvee L_2'' = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.0 & 0.0 \\ y_2 & 0.8 & 0.0 \\ y_3 & 0.2 & 1.0 \end{matrix}.$$

It is easily seen that $\{R\} = \mathcal{R} \cap \mathbf{L}$. Thus $\mathcal{R} \neq \emptyset$ and obviously R is the unique minimal element (i.e. the minimum) of $\mathcal{R}$. Note that $\mathcal{R}$ has not the greatest element.

We conclude this Section recalling the papers of Czogala, Drewniak and Pedrycz [3], Di Nola [4], Drewniak [11, 12], Higashi and Klir [17], Luo Cheng Zhong [19], Miyakoshi and Shimbo [21], Pappis and Sugeno [23], Sanchez [26], Wang and Yuan [30] and the authors of [32] where a detailed discussion on the lower solutions of a max-t_1 fuzzy equation is presented and see [9] for max- $t^{(p)}$ fuzzy equations, where $p \geq 1$.

6. General results and Boolean solutions.

In this Section we consider a general norm t without requiring particular assumptions on it. Following Sanchez [26], we recall that the binary operation "σ" defined as $a\sigma b = b$ if $a \geq b$, $a\sigma b = 0$ if $a < b$ and as in [10], considering the Equation (3.2), we put for any $j \in \mathbf{I}_m$:

$$\Gamma_j = \{i \in \mathbf{I}_n : A_i \geq B_j\}.$$

Further, as in [8], we define the m sets:

$$\Gamma_j^* = \{i \in \mathbf{I}_n : A_i = 1\}$$

for any $j \in \mathbf{I}_m$. Note that $\Gamma_j = K_j$ for any $j \in \mathbf{I}_m$ if t is a norm satisfying property (2.7). We now prove that

Theorem 6.1. Let $\Gamma_j^* \neq \emptyset$ for any $j \in \mathbf{I}_m$. Then the fuzzy relation $A\Phi B \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ pointwise defined as $(A\Phi B)_{ij} = A_i \sigma B_j$ for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, belongs to $\mathcal{M}$.

Proof. We have for any $j \in \mathbf{I}_m$:

$$\bigwedge_{i=1}^{n}[A_i t(A_i \sigma B_j)] = \{\bigwedge_{i\in\Gamma_j}[A_i t(A_i\sigma B_j)]\}\bigwedge\{\bigwedge_{i\notin\Gamma_j}[A_i t(A_i\sigma B_j)]\} =$$

$$\{\bigwedge_{i\in\Gamma_j}(A_i t B_j)\}\bigwedge\{\bigwedge_{i\notin\Gamma_j}(A_i t 0)\} = \bigwedge_{i\in\Gamma_j}\{(A_i t B_j)\}\bigwedge 0 =$$

$$\{\bigwedge_{i\in\Gamma_j-\Gamma_j^*}(A_i t B_j)\}\bigwedge\{\bigwedge_{i\in\Gamma_j}(1 - tB_j)\} = \{\bigwedge_{i\in\Gamma_j-\Gamma_j^*}(A_i t B_j)\}\bigwedge B_j = B_j$$

since $A_i t B_j \leq 1 t B_j = B_j$ for any $i \in \Gamma_j - \Gamma_j^*$. This implies that $(A\Phi B)$ belongs to $\mathcal{M}$.

This theorem was proved in [9] for $t = t^{(p)}$, $p \geq 1$. In [10], the following result was shown for norms satisfying property (2.7).

Theorem 6.2. If $\mathcal{M} \neq \emptyset$, then $\Gamma_j \neq \emptyset$ for any $j \in \mathbf{I}_m$.

We observe that the property (2.7) is not crucial in the proof of the above result. Hence Theorem 6.2 holds for any norm t and we now show an analogous theorem for the Equation (3.1), but we first need to define the sets

$$\Gamma_{ik} = \{j \in \mathbf{I}_m : Q_{ij} \geq T_{ik}\}$$

for any $i \in \mathbf{I}_n$, $k\mathbf{I}_p$. Then we have that

Theorem 6.3. If $\mathcal{R} \neq \emptyset$, then $\Gamma_{ik} \neq \emptyset$ for any $i \in \mathbf{I}_n$, $k \in \mathbf{I}_p$.

Proof. If $\mathcal{R} \neq \emptyset$, then $\mathcal{M}_h \neq \emptyset$ for any $h \in \mathbf{I}_n$ by (3.4). Hence $\Gamma_{ik} \neq \emptyset$ for any $i \in \mathbf{I}_n$, $k \in \mathbf{I}_p$ by Theorem 6.2.

This theorem was proved in [28] for $t = t_1$. Theorem 6.3 is not invertible as it is proved in the following example [28].

Example 7. Let $t = t_4$, $m = n = p = 3$, $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ and $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$ be given by

$$Q = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 \\ x_2 \\ x_3 \end{matrix}\hspace{-3.5em}\begin{matrix} \\ \begin{pmatrix} 0.0 & 0.9 & 1.0 \\ 0.2 & 0.0 & 0.8 \\ 0.5 & 0.0 & 0.9 \end{pmatrix}\end{matrix} \qquad T = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 \\ x_2 \\ x_3 \end{matrix}\hspace{-3.5em}\begin{matrix} \\ \begin{pmatrix} 0.8 & 0.3 & 0.6 \\ 0.6 & 0.5 & 0.8 \\ 0.6 & 0.5 & 0.9 \end{pmatrix}\end{matrix}$$

We have

$$Q^{-1}\Psi_4 T = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 \\ y_2 \\ y_3 \end{matrix}\hspace{-3.5em}\begin{matrix} \\ \begin{pmatrix} 0.6 & 0.5 & 0.9 \\ 0.8 & 0.7 & 1.0 \\ 0.7 & 0.6 & 1.0 \end{pmatrix}\end{matrix}$$

and $\Gamma_{11} = \Gamma_{12} = \Gamma_{13} = \{2,3\}$, $\Gamma_{21} = \Gamma_{22} = \Gamma_{23} = \Gamma_{31} = \Gamma_{33} = \{3\}, \Gamma_{32} = \{1,3\}$, i.e. $\Gamma_{ik} \neq \emptyset$ for any $i, k \in \mathbf{I}_3$, but

$$(Q^{-1}\Psi_4 T) = \perp_{t_4} Q \neq T$$

since

$$T_{22} = 0.5 > 0.4 = 0 \bigwedge 0 \bigwedge 0.4 = \bigwedge_{j=1}^{3} [Q_{ij} t_4 (Q^{-1}\Psi_4 T)_{jk}].$$

Thus $\mathcal{R} \neq \emptyset$ by Theorem 4.1.

The rest of the present paper is dedicated to the study of Boolean solutions of the Equations (3.1) and (3.2). We define Boolean solution of the Equation (3.1) (resp. (3.2)) a fuzzy relation $R \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z})$ (resp. $M \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$) such that $R_{jk} \in \{0,1\}$ (resp. $M_{ij} \in \{0,1\}$) for any $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$ (resp. $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$). The following result holds for the Equation (3.2) under any norm t.

Theorem 6.4. The Equation (3.2) has a Boolean solution iff for any $j \in \mathbf{I}_m$ either exists at least an index $i \in \mathbf{I}_n$ such that $A_i = B_j$ if $B_j > 0$ or $B_j = 0$.

Proof. The condition is necessary. Indeed, let $M \in \mathcal{M}$ be a Boolean solution of the Equation (3.2). For any $j \in \mathbf{I}_m$, we define the following sets:

$$M_0^j = \{i \in \mathbf{I}_n : M_{ij} = 0\} \text{ and } \mathcal{M}_1^j = \{i \in \mathbf{I}_n : M_{ij} = 1\}.$$

Then, if $B_j > 0$, we have

$$B_j = \bigvee_{i=1}^{n} (A_i t M_{ij}) = \{ \bigvee_{i \in M_0^j} (A_i t 0)\} \bigvee \{ \bigvee_{i \in M_1^j} (A_i t 1)\} = \bigvee_{i \in M_1^j} A_i$$

and hence there exists at least an index $i \in \mathcal{M}_1^j$ such that $A_i = B_j$.

The condition is sufficient. To show this, we define the fuzzy relation

$$M_{ij} = \begin{cases} 1 & \text{if } i = i^*,\ B_j > 0, \\ 0 & \text{if } i \leq i^*,\ B_j > 0 \text{ or } B_j = 0. \end{cases}$$

Then we have if $B_j > 0$,

$$\bigvee_{i=1}^{n} (A_i t M_{ij}) = \{ \bigvee_{i=i^*} (A_i t M_{ij})\} \bigvee (A_{i^*} t M_{i^* j}) = 0 \bigvee (B_j t 1) = 0 \bigvee B_j = B_j.$$

If $B_j = 0$, then

$$\bigvee_{i=1}^{n} (A_i t M_{ij}) = \bigvee_{i=1}^{n} (A_i t 0) = 0 = B_j.$$

This means that M is a Boolean element of $\mathcal{M}$.

Theorem 6.4 was obtained in [6] for $t = t_1$ and in [9] for $t = t_3^{(p)}$, where $p \geq 1$. Similarly, it can be shown the following result for the Equation (3.1).

Theorem 6.5. The Equation (3.1) has a Boolean solution iff for any $i \in \mathbf{I}_n$, $k \in \mathbf{I}_p$, there exists at least an index $j \in \mathbf{I}_m$ such that either $Q_{ij} = T_{ik}$ if $T_{ik} > 0$ or $T_{ik} = 0$.

If the Equation (3.1) (resp. (3.2)) has not Boolean solutions and if $\mathcal{R} \neq \emptyset$ (resp. $\mathcal{M} \neq \emptyset$), then we apply an algorithm of Di Nola and Ventre [7] in order to find the maximal Booleanity, i.e. the greatest number of 0 and 1, present in a solution $R \in \mathcal{R}$ (resp. $M \in \mathcal{M}$). As pointed out in [7], we note that the problem of maximizing the Booleanity of an element $R \in \mathcal{R}$ (resp. $M \in \mathcal{M}$) does not coincide with the problem of minimizing the entropy of R (resp. M) since the amount of entropy depends from the non-Boolean entries of R (resp. M). By other hand, this problem has been solved in [8] for max-min fuzzy equations and in [10] for max-t fuzzy equations under a norm satisfying property (2.7).

7. Boolean solutions under LSC norms.

In this Section, we consider fuzzy Equations (3.1) and (3.2) under LSC norms. If $\mathcal{R} \neq \emptyset$, we denote by $\mathbf{B} = \mathbf{B}(Q,T)$ the subset of $\mathcal{R}$ constituted by the Boolean elements of $\mathcal{R}$. In accordance to Sessa [29], we define the following element $S \in \mathbf{B}$:

$$S_{jk} = \begin{cases} 1 & \text{if } (Q^{-1}\Psi T)_{jk} = 1, \\ 0 & \text{otherwise}, \end{cases}$$

for any $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$. Then Theorem 6.5 can be formulated in another interesting version.

Theorem 7.1. Let $\mathcal{R} \neq \emptyset$. Then $\mathbf{B} \neq \emptyset$ iff $S \in \mathbf{B}$ and further $S \geq R$ for any $R \in \mathbf{B}$.

Proof. Of course, we only prove the non-trivial implication. Let $R \in \mathbf{B} \neq \emptyset$. Since Theorem 4.1 implies that $(Q^{-1}\Psi T) \in \mathcal{R}$, then we have $R \leq S \leq (Q^{-1}\Psi T)$ for any $R \in \mathbf{B}$. Therefore Lemma 3.1 assures the thesis.

This theorem was established in [29] for $t = t_1$ but in the general context of bounded Brouwerian lattices, in accordance to Sanchez [25]. In other words, Theorem 7.1 guarantees that S, when it exists, is the greatest element of $\mathbf{B}$.

Example 8. Let $t = t_2$, $m = n = p = 3$, $Q \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ and $T \in \mathbf{F}(\mathbf{X} \times \mathbf{Z})$ be given by

$$Q = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.8 & 0.7 & 0.5 \\ x_2 & 0.6 & 0.3 & 0.5 \\ x_3 & 0.4 & 0.0 & 0.2 \end{matrix}, \quad T = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.7 & 0.8 & 0.5 \\ x_2 & 0.5 & 0.6 & 0.5 \\ x_3 & 0.2 & 0.4 & 0.2 \end{matrix},$$

We have

$$Q^{-1}\psi_2 T = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.5 & 1.0 & 0.5 \\ y_2 & 1.0 & 1.0 & 0.7 \\ y_3 & 1.0 & 1.0 & 1.0 \end{matrix}, \quad S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.0 & 1.0 & 0.0 \\ y_2 & 1.0 & 1.0 & 0.0 \\ y_3 & 1.0 & 1.0 & 1.0 \end{matrix},$$

and it is easily seen that

$$S \bot_{t_2} Q = T.$$

Theorem 7.2. If for any $i \in \mathbf{I}_n$, $k \in \mathbf{I}_p$,

$$\bigvee_{j=1}^{m} Q_{ij} = T_{ik},$$

then $\mathcal{R} \neq \emptyset$ and $(Q^{-1}\Psi T)_{jk} = S_{jk} = 1$ for any $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$.

Proof. We observe that for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, $k \in \mathbf{I}_p$:

$$Q_{ij} \leq \bigvee_{j=1}^{m} Q_{ij} = T_{ik}$$

and this implies, by Definition 3.5 and property (2.8), that

$$(Q^{-1}\Psi T)_{jk} = \bigwedge_{i=1}^{n} (Q_{ij} \psi T_{ik}) = \bigwedge_{i=1}^{n} 1 = 1 = S_{jk}.$$

Since

$$T_{ik} = \bigvee_{j=1}^{m} Q_{ij} = \bigvee_{j=1}^{m} (Q_{ij} t 1) = \bigvee_{j=1}^{m} [Q_{ij} t (Q^{-1}\Psi T)_{jk}]$$

for any $i \in \mathbf{I}_n$, $k \in \mathbf{I}_p$, we have $\mathcal{R} \neq \emptyset$ by Theorem 4.1.

Denoting by $\mathbf{B}' = \mathbf{B}(A, B)$ the subset of $\mathcal{M}$ constituted by the Boolean elements $M \in \mathcal{M}$, one can enunciate, similarly to Theorem 7.1, the following result for the Equation (3.2).

Theorem 7.3. Let $\mathcal{M} \neq \emptyset$. Then $\mathbf{B}' \neq \emptyset$ iff the fuzzy relation $S' \in \mathbf{F}(\mathbf{X} \times \mathbf{Y})$ pointwise defined as

$$S'_{jk} = \begin{cases} 1 & \text{if } (A\Psi B)_{ij} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for any $i \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, belongs to $\mathbf{B}'$ and further $S' \geq M$ for any $M \in \mathbf{B}'$.

Let $\mathbf{B}'_h = \mathbf{B}(Q_h, T_h)$, $h \in \mathbf{I}_n$, be the set of the Boolean solutions of the Equation (3.3). Of course, it follows that

$$\mathbf{B} = \mathbf{B}'_1 \cap \mathbf{B}'_2 \cap \ldots \cap \mathbf{B}'_n, \tag{7.1}$$

If S'_h, $h \in \mathbf{I}_n$, stands for the greatest element of $\mathbf{B}'_h$, then we have the following theorem.

Theorem 7.4. If $\mathbf{B} \neq \emptyset$, then

$$S = \bigwedge_{h=1}^{n} S'_h.$$

Proof. The fuzzy relation S belongs to each set $\mathbf{B}'_h$, $h \in \mathbf{I}_n$, since (7.1) holds. Thus $S \leq S_h$ for any $h \in \mathbf{I}_n$ by Theorem 7.3. Then

$$S \leq W = \bigwedge_{h=1}^{..} S'_h \leq S'_h \tag{7.2}$$

for any $h \in \mathbf{I}_n$. By Lemma 3.1, the fuzzy relation W belongs to the set $\mathcal{M}_h$ for any $h \in \mathbf{I}_n$. Since (7.1) holds, W is in $\mathbf{B}$ and hence $W \leq S$ by Theorem 7.1.

The converse inequality is in (7.2) and then $W = S$.

Example 9. Recalling Example 8, we have

$$Q_1\Psi_2T_1 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.875 & 1.000 & 0.625 \\ 1.000 & 1.000 & 0.714 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \end{array},$$

$$Q_2\Psi_2T_2 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.833 & 1.000 & 0.833 \\ 1.000 & 1.000 & 1.000 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \end{array},$$

$$Q_3\Psi_2T_3 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.500 & 1.000 & 0.500 \\ 1.000 & 1.000 & 1.000 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \end{array},$$

Therefore

$$S'_1 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.000 & 1.000 & 0.000 \\ 1.000 & 1.000 & 0.000 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \end{array} \quad S'_2 = S'_3 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.000 & 1.000 & 0.000 \\ 1.000 & 1.000 & 1.000 \\ 1.000 & 1.000 & 1.000 \end{pmatrix} \end{array},$$

and it is easily seen that the fuzzy relation S of Example 8 is equal to $S'_1 \bigwedge S'_2 \bigwedge S'_3$.

8. Minimal Boolean solutions of the Equation (3.2).

Although the results of this Section can be deduced easily from [29], we give them for sake of completeness. Here we characterize the minimal Boolean solutions of the Equation (3.2) under a norm t satisfying property (2.7), assuming of course $\mathbf{B}' \neq \emptyset$. In accordance to Di Nola and Ventre [7], we define in $\mathcal{M}$ the following binary operation:

$$(M\Delta M')_{ij} = \begin{cases} M'_{ij} & \text{if } M_{ij} \neq \emptyset, \\ 0 & \text{if } M_{ij} = 0, \end{cases}$$

for any $h \in \mathbf{I}_n$, $j \in \mathbf{I}_m$, M, $M' \in \mathcal{M}$. Remembering that a norm satisfying property (2.7) is necessarily LSC and USC, we have that $S' \in \mathbf{B}'$ by Theorem 7.3 and we define the following set:

$$\mathbf{S} = \mathbf{S}(A, B) = \{L \in \mathcal{M} : L \leq S'\},$$

where L is a minimal element of $\mathcal{M}$. **S** is nonempty by Theorem 5.5 and clearly we have $L \leq (L\Delta S') \leq S'$ for any $L \in \mathbf{S}$. By Lemma 3.1, the fuzzy relation $(L\Delta S')$ belongs to $\mathcal{M}$ and it is obviously an element of $\mathbf{B}'$. Then the following set:

$$\mathbf{S}_\delta = \mathbf{S}_\delta(A,B) = \{L\Delta S' : L \in \mathbf{S}\}$$

is a subset of $\mathbf{B}'$ and the following result of [29] holds.

Theorem 8.1. If $\mathbf{B}' \neq \emptyset$, the minimal elements of $\mathbf{B}'$ are the minimal elements of $\mathbf{S}_\delta$ and vice versa.

We illustrate Theorem 8.1 with a suitable numerical example assuming $t = t_1$ and observing that (cfr. Example 2 of [10]) for $a = b$, we have $\mathbf{G}_t(a,b) = [a,1]$ and $\mathbf{G}_t(a,b) = \{b\}$ for $a > b$, $a, b \in [0,1]$.

Example 10. Let $m = 4$, $n = 3$, $A \in \mathbf{F}(\mathbf{X})$ and $B \in \mathbf{F}(\mathbf{Y})$ be given by

$$A = \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \begin{pmatrix} 0.6 & 0.4 & 0.7 \end{pmatrix}, \quad B = \begin{matrix} y_1 & y_2 & y_3 & y_4 \end{matrix} \begin{pmatrix} 0.4 & 0.6 & 0.0 & 0.7 \end{pmatrix}.$$

We have that

$$A\Psi_1 B = \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 1.0 & 0.0 & 1.0 \\ x_2 & 1.0 & 1.0 & 0.0 & 1.0 \\ x_3 & 0.4 & 0.6 & 0.0 & 1.0 \end{matrix}, \quad S' = \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.0 & 1.0 & 0.0 & 1.0 \\ x_2 & 1.0 & 1.0 & 0.0 & 1.0 \\ x_3 & 0.0 & 0.0 & 0.0 & 1.0 \end{matrix},$$

$$A\Theta_1 B = \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0.6 & 0.0 & 0.0 \\ x_2 & 0.4 & 0.0 & 0.0 & 0.0 \\ x_3 & 0.4 & 0.6 & 0.0 & 0.7 \end{matrix}.$$

Since $S' \perp_{t_1} A = B$, we have $\mathbf{B}' \neq \emptyset$ and by Theorem 5.3, there exist six lower solutions given by

$$L_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 \end{pmatrix} \end{array}, \quad L_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.0 & 0.6 & 0.0 & 0.0 \\ 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 \end{pmatrix} \end{array},$$

$$L_3 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.0 & 0.6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.0 & 0.0 & 0.7 \end{pmatrix} \end{array}, \quad L_4 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.7 \end{pmatrix} \end{array},$$

$$L_5 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.7 \end{pmatrix} \end{array}, \quad L_6 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.6 & 0.0 & 0.7 \end{pmatrix} \end{array},$$

and it is easily seen that $\mathbf{S} = L_2$. Hence $\mathbf{S}_\delta \in= \{L_2 \Delta S'\}$, where

$$L_2 \Delta S' = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \end{array},$$

It is the unique minimal element of $\mathbf{B}'$ and every Boolean solution lies between $(L_2 \Delta S')$ and S'.

Theorem 8.2. If $\mathbf{B}' \neq \emptyset$, for any $M \in \mathbf{B}'$ there exists an element $M' \in \mathbf{S}_\delta$ such that $M' \leq M$.

Proof. By Theorem 5.3, for any $M \in \mathbf{B}'$ there exists a minimal element $L \in \mathcal{M}$ such that $L \leq M \leq S'$. If $M_{ij} = 0$, we have $L_{ij} = 0$ and hence $(L\Delta S')_{ij} = 0 = (L\Delta M)_{ij}$. If

$M_{ij} = 1$, we have $S_{ij} = 1$ and hence $(L\Delta M)_{ij} = M_{ij} = 1 = S_{ij} = (L\Delta S')_{ij}$. By setting $(L\Delta S') = M' \in \mathbf{S}_\delta$, then we deduce $M' = (L\Delta S') = (L\Delta M) \leq M$, i.e. the thesis.

Lemma 8.3. Let M, M', $M" \in \mathcal{M}$. Then

$$(M'\Delta M) \bigvee (M"\Delta M) = (M' \bigvee M")\Delta M.$$

Proof. Trivial since it follows directly from the definition of the operator δ.

Theorem 8.4.

$$\bigvee (L\Delta S') = (A\Theta B)\Delta S',$$

where the sup is calculated on the minimal elements of $\mathcal{M}$.

Proof. By Theorem 5.4, $A\Theta B$ is the fuzzy union of all the minimal elements of $\mathcal{M}$. We have by Lemma 8.3,

$$(A\Theta B)\Delta S' = (\bigvee L)\Delta S' = \bigvee (L\Delta S'),$$

i.e. the thesis.

9. Minimal Boolean solutions of the Equation (3.1).

In [29], the author proved directly Theorem 8.1 for the Equation (3.1) assuming $t = t_1$ in a linear lattice. However, with a slight different presentation from [29], we like to show here how Theorem 8.2 concerning Equation (3.2) can be applied to the Equation (3.1), for which we assume $\mathbf{B} \neq \emptyset$.

Bearing in mind Equation (3.3) and similarly to the symbology of Section 8, we define the sets:

$$\mathbf{S}_h = \mathbf{S}(Q_h, T_h) = \{L_h \in \mathbf{L}_h : L_h \leq S'_h\}$$

$$\mathbf{S}_\delta^h = \mathbf{S}_\delta(Q_h, T_h) = \{L_h \Delta S_h : L_h \in \mathbf{S}_h\}$$

for any $h \in \mathbf{I}_n$, where L_h is a minimal element of $\mathcal{M}_h$. By (7.1), the greatest element S of $\mathbf{B}$ is in each set $\mathbf{B}'_h$ and by Theorem 8.2, the set

$$\Lambda_h = \{M_h \in \mathbf{S}^h_\delta : M_h \leq S\} \neq \emptyset$$

for any $h \in \mathbf{I}_n$. If we define the set

$$\Lambda = \Lambda(Q,T) = \{U \in \mathbf{F}(\mathbf{Y} \times \mathbf{Z}) : U = \bigvee_{h=1}^{n} M_h, M_h \in \mathbf{S}^h_\delta\},$$

we have that $M_h \leq U \leq S$ for any $h \in \mathbf{I}_n$ and $U \in \Lambda$. This means that $U \in \mathcal{M}_h$ for any $h \in I_n$ by Lemma 3.1 and hence $U \in \mathcal{R}$ since (3.4) holds. Of course, the fuzzy relation U, that is fuzzy union of Boolean matrices, is also an element of $\mathbf{B}$. Hence $\Lambda \subseteq \mathbf{B}$. We now prove the following characterization theorem.

Theorem 9.1. The minimal elements of $\mathbf{B}$ are the minimal elements of Λ and vice versa.

Proof. Let U be minimal in Λ and let $W \in \mathbf{B}$ be such that $W \leq U$. We must prove that $W = U$. Since $W \in \mathbf{B}'_h$ for any $h \in \mathbf{I}_n$, by Theorem 8.2, let $M_h \in \mathbf{S}^h_\delta$ be such that $M_h \leq W$ for any $h \in \mathbf{I}_n$. Then

$$M = \bigvee_{h=1}^{n} M_h \leq W \leq U,$$

but U is minimal in Λ and this implies $W \geq M = U \geq W$, i.e. $W = U$. Vice versa, it suffices to prove only that a minimal element U of $\mathbf{B}$ is in Λ. Indeed, since $U \in \mathbf{B}'_h$ for any $h \in \mathbf{I}_n$, by Theorem 8.2, let $M_h \in \mathbf{S}^h_\delta$ be such that $M_h \leq U$ for any $h \in \mathbf{I}_n$. Then

$$M_h \leq M = \bigvee_{h=1}^{n} M_h \leq U$$

and hence the fuzzy relation M is in $\mathbf{B}'_h$ for any $h \in \mathbf{I}_n$ by Lemma 3.1. Since (7.1) holds, M is in $\mathbf{B}$ and therefore $M = U$ since U is minimal in $\mathbf{B}$. It is evident that M is in Λ.

We illustrate Theorem 9.1 with the following example.

Example 10. Let $t = t_2$ and pointing out that $\mathbf{G}_t(a, b) = \{1\}$ if $a = b > 0$ and $\mathbf{G}_t \in (a, b) = \{b/a\}$ if $a > b > 0$, we have from Example 9,

$$Q_1\Theta_2T_1 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.875 & 1.000 & 0.625 \\ 1.000 & 0.000 & 0.714 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} \end{array},$$

$$Q_2\Theta_2T_2 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.833 & 1.000 & 0.833 \\ 0.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 1.000 \end{pmatrix} \end{array},$$

$$Q_3\Theta_2T_3 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.500 & 1.000 & 0.500 \\ 0.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 1.000 \end{pmatrix} \end{array},$$

where Θ_2 stands for Θ_{t_2}. Bearing in mind S_1', S_2', S_3' calculated in Example 9, we have that $\mathbf{S}_1 = \{L\}$, $\mathbf{S}_2 = \mathbf{S}_3 = \{L'\}$, where

$$L = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.000 & 1.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} \end{array}, \quad L' = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 1.000 \end{pmatrix} \end{array}.$$

In this case, we have $L\Delta S_1 = L \leq S$ and $L'\Delta S_2 = L'\Delta S_3 = L' \leq S$, where S is given in Example 8. Thus $\Lambda_1 = \mathbf{S}_\delta^1 = \{L\}$, $\Lambda_2 = \Lambda_3 = \mathbf{S}_\delta^2 = \mathbf{S}_\delta^3 = \{L'\}$, hence $\Lambda = \{M\}$, where

$$M = L \bigvee L' = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.000 & 1.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 1.000 \end{pmatrix} \end{array}.$$

M is the unique minimal element of **B** and every Boolean solution lies between M and S.

Another numerical example for $t = t_1$ can be found in [29].

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