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LATTICE VALUED ALGEBRAS

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1. Introduction.

The concept of fuzzy set, introduced by L.A. Zadeh [14], was applied to the theory of groupoids and groups by A. Rosenfeld [12] that defines the concept of fuzzy-groupoid. Subsequently, his definition, that was generalized by C.V. Negoita and D.A. Ralescu [11], gives rise to a very large literature. Applications to code theory can be find in [1] and [7]. Indeed, a free (pure, very pure, unitary) fuzzy subsemigroup defines a class of free (pure, very pure, unitary) subsemigroup via its cuts. Consequently, the tecniques proposed in the quoted papers to building up fuzzy subsemigroups of the required type, enable us also to obtain a large class of subsemigroup of the same type.

In this paper we propose a general approach to the theory of fuzzy algebras, while the early existing papers deal with a particular type of fuzzy structures as fuzzy groups, fuzzy ideals, fuzzy vectorial spaces and so on. Such approach is strictly related to the algebraical treatment of nonclassical logic as devised by H. Rasiowa, R. Sikorski and others. Indeed, we are convinced that this is the natural theoretical framework for fuzzy set theory (see [2] and [6]).

Namely, we propose for fuzzy algebras some basic tools of universal algebra type, such as the concepts of homomorphism, congruence, quotient, direct product, reduced product

2. Lattice valued algebras.

Let X be a set and L a complete lattice, then the direct product L^X is a complete lattice whose elements are named L-subsets of X [8]. The join and meet operations are called union and intersection. The infimum and supremum elements of L^X are denoted by f_0 and f_1, respectively. We denote by (X, f, L) an element f of L^X.

If $A = (D_A, F_A)$ is an algebra with domain D_A and operation set F_A, then we denote by (A, a, L) an L-subset of D_A; such an L-subset is called lattice valued algebra or $\mathcal{L}$-algebra on A if:

$$a(s(x_1, \cdots, x_n)) \geq a(x_1) \wedge \cdots \wedge a(x_n) \tag{1}$$

for every n-ary operation $s \in F_A$ and $x_1, \cdots, x_n \in D_A$.

Every constant function $a : D_A \to L$ is an $\mathcal{L}$-algebra, in particular (A, f_1, L) and (A, f_0, L) are $\mathcal{L}$-algebras. If (A, a, L) is an $\mathcal{L}$-algebra then an $\mathcal{L}$-subalgebra of (A, a, L) is an $\mathcal{L}$-algebra (A, a', L) such that $a' \leq a$. Of course, every $\mathcal{L}$-algebra on A is an $\mathcal{L}$-subalgebra of (A, f_1, L). Since we can identify (A, f_1, L) with A, an $\mathcal{L}$-algebra on A is also named $\mathcal{L}$-subalgebra of A. It is easy to prove that an L-subset (A, a, L) of A is an $\mathcal{L}$-algebra if and only if every α-cut $C_a^\alpha = \{x \in A \colon a(x) \geq \alpha\}$ is a subalgebra of A. Then to every $\mathcal{L}$-algebra (A, a, L_a) it is associated a family $(C_a^\alpha)_{\alpha \in L}$ of subalgebras of A. Obviously, if $\alpha, \beta \in L$ and $\alpha \leq \beta$ then $C_a^\alpha \supseteq C_a^\beta$. When $L = \{1\}$ is the one element lattice we can identify an $\mathcal{L}$-algebra (A, a, L) with the algebra A. Then the $\mathcal{L}$-algebra concept generalizes the usual concept of algebra. When $L = \{0, 1\}$ is the two elements lattice and $a \colon A \to L$, then (A, a, L) is an $\mathcal{L}$-algebra if and only if a is the characteristic function of a subalgebra of A. Thus the $\mathcal{L}$-algebra concept generalizes the concept of subalgebra.

The following theorem shows that the class of all $\mathcal{L}$-subalgebras of a given $\mathcal{L}$-algebra is a closure system. In particular, the class of all $\mathcal{L}$-algebras on a given algebra A is a closure system and to every L- subset (A, a, L) of A we can associate the $\mathcal{L}$-subalgebra on A generated by (A, a, L).

Theorem 2.1. The intersection of a family of $\mathcal{L}$-subalgebras of an $\mathcal{L}$-algebra (A, a, L) is an $\mathcal{L}$-subalgebra of (A, a, L).

Proof. Let $(A, a_i, L)_{i\in I}$ be a family of $\mathcal{L}$-subalgebras of (A, a, L) then $\bigwedge_{i\in I} a_i \leq a$. Moreover from $a_i(s(x_1, \cdots, x_n)) \geq a_i(x_1) \wedge \cdots \wedge a_i(x_n)$ for every $i \in I$, it follows that

$$\bigwedge_{i\in I} a_i(s(x_1), \cdots, x_n)) \geq \bigwedge_{i\in I}(a_i(x_1)\wedge, \cdots, \wedge a_i(x_n)) =$$

$$(\bigwedge_{i\in I} a_i(x_1)) \wedge \cdots \wedge (\bigwedge_{i\in I} a_i(x_n)).$$

3. The category of $\mathcal{L}$-algebras.

Define the type τ of an $\mathcal{L}$-algebra (A, a, L) as the type of the algebra A (see [9]). Then we will show that the class of all $\mathcal{L}$-algebras of a given type τ define a category $\mathcal{C}(\tau)$ in a very natural manner. The objects of $\mathcal{C}(\tau)$ are the $\mathcal{L}$-algebras of type τ. The morphisms from an $\mathcal{L}$-algebra (A, a, L_a) to an $\mathcal{L}$-algebra (B, b, L_b) are pairs (h, k), with $h\colon A \to B$ homomorphism from A to B and $k\colon L_a \to L_b$ homomorphism from L_a to L_b, such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ L_A & \xrightarrow{k} & L_B \end{array}$$

that is $ak = hb$. A morphism is also named $\mathcal{L}$-homomorphism. The product of two morphisms (h, k) and (h', k') is defined by setting $(h, k) \circ (h', k') = (h \circ h', k \circ k')$. It is easy to prove that the product of two morphisms is a morphism and that the identity is a morphism.

The category $\mathcal{C}(\tau)$ is in some relation with the product of the category of the algebras of type τ and the category of the complete lattice. Namely, if we associate to every $\mathcal{L}$-algebra (A, a, L) the pair (A, L) and to every morphism (h, k) still (h, k), then we obtain a forgetful functor.

Theorem 3.1. $\mathcal{C}(\tau)$ has direct products. Namely if $((A_i, a_i, L_i))_{i\in I}$ is a family of $\mathcal{L}$-algebras, $A = \prod A_i$, $L = \prod L_i$ and $a\colon A \to L$ is defined by setting $a((x_i)_{i\in I}) = (a_i(x_i))_{i\in I}$ for every $(x_i)_{i\in I} \in A$, then (A, a, L) is the direct product of $((A_i, a_i, L_i))_{i\in I'}$.

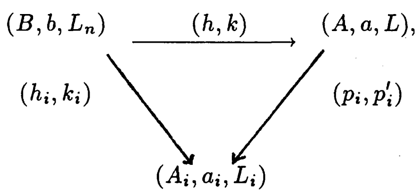
Proof. Let h be an n-ary operation of A and $x_1 = (x_i^1)) =_{i\in I'} \cdots, x_n = (x_i^n)_{i\in I}$ elements of A. Then $h(x_1, \cdots, x_n) = (h_i(x_i^1, \cdots, x_i^n))_{i\in I}$ where h_i is a suitable operation of A_i, for every $i \in I$. It follows that

$$a(h(x_1, \cdots, x_n)) = (a_i(h_i(x_i^1, \cdots, x_i^n)))_{i\in I}$$

$$\geq (a_i(x_i^1) \wedge \cdots \wedge a_i(x_i^n))_{i\in I} = a(x_1) \wedge \cdots \wedge a(x_n).$$

This proves that (A, a, L) is an $\mathcal{L}$-algebra.

Let us assume now that (B, b, L_b) is an $\mathcal{L}$-algebra and that, for every $i \in I$, (h_i, k_i) is a morphism from (B, b, L_b) to (A_i, a_i, L_i). We have to prove that there exists a unique morphism (h, k) such that the diagram



(2)

is commutative for every $i \in I$. Let us now define h by setting $h(x) = (h_i(x))_{i\in I}$ for every $x \in B$ and k by setting $k(y) = (k_i(y))_{i\in I}$ for every $y \in L_b$. It is obvious that h and k are homomorphisms from B to A and from L_b to L respectively. Furthermore, for every $x \in B$ it is

$$k(b(x)) = (k_i(b(x)))_{i\in I} = (a_i(h_i(x)))_{i\in I} = a(h(x))$$

and this proves that (h, k) is a morphism. It is obvious that (2) commutes. In order to prove the unicity, it is enough to observe that if (h', k') is a morphism such that $p_i'(k'(y)) = k_i(y)$ for every $x \in B$ and $y \in L_b$, then $h' = h$ and $k' = k$.

4. Congruences and quotients.

An $\mathcal{L}$-congruence on an $\mathcal{L}$-algebra (A, a, L) is a pair (θ, ψ) of congruences respectively of A and L, such that, for every $x, y \in A$,

$$x \equiv_\theta y \text{ implies } a(x) \equiv_\psi a(y). \tag{3}$$

The quotient of (A, a, L) by (θ, ψ) is the tern $(A/\theta, a', L/\psi)$ where a' is defined by setting $a'([x]_\theta) = [a(x)]_\psi$ for every $x \in A$.

Theorem 4.1. Every quotient of an $\mathcal{L}$-algebra is an $\mathcal{L}$- algebra.

Proof. Let $h \in F_A$ be an n-ary operation of A, h' the operation induced on A/θ by h and $[x_1]_\theta, \cdots, [x_n]_\theta$ elements of A/θ. Then

$$\begin{aligned} a'\left(h'\left([x_1]_\theta, \cdots, [x_n]_\theta\right)\right) &= a'\left([h(x_1, \cdots, x_n)]_\theta\right) \\ &= \left[a\left(h(x_1, \cdots, x_n)\right)\right]_\psi \geq \left[a(x_1) \wedge \cdots \wedge a(x_n)\right]_\psi \\ &= \left[a(x_1)\right]_\psi \wedge \cdots \wedge \left[a(x_n)\right]_\psi \\ &= a'\left([x_1]_\theta\right) \wedge \cdots \wedge a'\left([x_n]_\theta\right). \end{aligned}$$

The following theorem shows that, as in the classical case, the $\mathcal{L}$- congruences are strictly related with the $\mathcal{L}$-homomorphisms.

Theorem 4.2. If (h, k) is an $\mathcal{L}$-homomorphism from (A, a, L_a) to (B, b, L_b) then the relations $\theta_h = \{(x, y) \in A^2 \colon h(x) = h(y)\}$ and $\psi_k = \{(u, v) \in L_a^2 \colon k(u) = k(v)\}$ define an $\mathcal{L}$- congruence of (A, a, L_a). Moreover every $\mathcal{L}$-congruence is of such type.

Proof. It is well known that θ_h and ψ_k are congruences of A and L_a respectively. Moreover from $x \equiv_{\theta_h} y$ it follows that $h(x) = h(y)$ and therefore that $b(h(x)) = b(h(y))$. Since $bh = ka$ we have also that $k(a(x)) = k(a(y))$ and $a(x) \equiv_{\psi_k} a(y)$. This proves that $(\theta_{h'}, \psi_k)$ is an $\mathcal{L}$-congruence.

Assume that (θ, ψ) is a congruence of (A, a, L_a) and let (B, b, L_b) be the quotient of (A, a, L_a). Then the canonical homomorphisms $h \colon A \to B$ and $k \colon L_a \to L_b$ determine an $\mathcal{L}$-homomorphism whose associated $\mathcal{L}$-conguence is (θ, ψ).

In the sequel an $\mathcal{L}$-homomorphism (h,k) such that both h and k are injective (surjective) is named $\mathcal{L}$-monomorphism ($\mathcal{L}$-epimorphism). If (h,k) is an $\mathcal{L}$-monomorphism and an $\mathcal{L}$-epimorphism, then (h,k) is named $\mathcal{L}$- isomorphism and the relative $\mathcal{L}$-algebras are called isomorphic.

Theorem 4.3. If (h,k) is an epimorphism from (A,a,L_a) to (B,b,L_b) then the quotient (A',a',L'_a) of (A,a,L_a) by $(\theta_{h'},\psi_k)$ is isomorphic to (B,b,L_b).

Proof. Let $h'\colon A' \to B$ and $k'\colon L'_a \to L'_b$ be defined by setting $h'([x]\theta_h) = h(v)$ and $k'([u]\psi_k) = k(u)$ for every $x \in A$ and $u \in L_a$. The functions h' and k' are isomorphisms between A/θ_h and B, and between L_a/ψ_k and L_b respectively. Moreover, for every $[x]\theta_h \in A/\theta_h$ it is

$$b(h'([x]_{\theta_h}) = b(h(x)) = k(a(x)) = k'([a(x)]_{\psi_k}) = k'(a'([x]_{\theta_h})).$$

This proves that (h',k') is the desired $\mathcal{L}$-isomorphism.

If (h,k) is an $\mathcal{L}$-homomorphism form (A,a,L_a) to (B,b,L_b) and (A,a',L_a) is a subalgebra of (A,a,L_a) then let us define the map $b'\colon \to L_b$ by setting, for every $z \in B$,

$$b'(z) = \begin{cases} \bigvee \{k(a'(x))\colon\ x \in h^{-1}(z)\} & \text{when } h^{-1}(z) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then the $\mathcal{L}$-subset (B,b',L_b) is named the image of (A,a,L_a) via (h,k). If $L_a = L_b$ and k is the identity then this definition coincides with the well known definition given in literature.

Theorem 4.4. If (h,k) is an $\mathcal{L}$-homomorphism from (A,a,L_a) to (B,b,L_b), with L_b infinitely distributive, and (A,a',L_a) is an $\mathcal{L}$-subalgebra of (A,a,L_a), then the image of (A,a',L_a) via (h,k) is an $\mathcal{L}$-subalgebra of (B,b,L_b).

Proof. Let us denote by (B,b',L_b) the image of (A,a',L_a) via (h,k), by $s \in F_A$ any operation of A and by $s' \in F_B$ the corrispondent operation of B. Then we will prove that, for every $y_1, \cdots, y_n$ belonging to B, it is

$$b'(s'(y_1, \cdots, y_n)) \geq b'(y_1) \wedge \cdots \wedge b'(y_n). \tag{4}$$

Now if, for a suitable $i = 1, \cdots, n$, $h^{-1}(y_1) = \emptyset$ then $a'(y_i) = 0$ and (4) is proved. Thus we can suppose that, for every $i = 1, \cdots, n$, there exists $x_i \in h^{-1}(y_i)$. Since

$$h(s(x_1, \cdots, x_n)) = s'(h(x_1), \cdots, h(x_n)) = s'(y_1, \cdots, y_n),$$

we have also that $h^{-1}(s'(y_1, \cdots, y_n)) \neq \emptyset$. Now by the hypothesis it is

$$a'(s(x_1, \cdots, x_n)) \geq a'(x_1) \wedge \cdots \wedge a'(x_n)$$

and therefore

$$k(a'(s(x_1, \cdots, x_n))) \geq k(a'(x_1)) \wedge \cdots \wedge k(a'(x_n)).$$

Then

$$\begin{aligned} b'(y_1) \wedge \cdots \wedge b'(y_n) &= \left[\bigvee \{k(a'(x_1))\colon x_1 \in h^{-1}(y_1)\}\right] \wedge \cdots \\ &\wedge \left[\bigvee \{k(a'(x_n))\colon x_n \in h^{-1}(y_n)\}\right] \\ &= \bigvee \{k(a'(x_1)) \wedge \cdots \wedge k(a'(x_n))\colon x_1 \in h^{-1}(y_1), \cdots, x_n \in h^{-1}(y_n)\} \\ &\leq \bigvee \{k(a'(s(x_1, \cdots, x_n)))\colon x_1 \in h^{-1}(y_1), \cdots, x_n \in h^{-1}(y_n)\} \\ &\leq \bigvee \{k(a'(s(x)))\colon x \in h^{-1}(s'(y_1, \cdots, y_n))\} = b'(s'(y_1, \cdots, y_n)). \end{aligned}$$

Finally in order to prove that $b' \leq b$ we observe that if y is an element of B such that $h^{-1}(y) = \emptyset$ then $b'(y) = 0 \leq b(y)$. If $h^{-1}(y) \neq \emptyset$ then

$$\begin{aligned} b'(y) &= \bigvee \{k(a'(x))\colon x \in h^{-1}(y)\} \\ &\leq \bigvee \{k(a(x))\colon x \in h^{-1}(y)\} \\ &\leq \bigvee \{b(h(x))\colon x \in h^{-1}(y)\} = b(y) \end{aligned}$$

To conclude this section, let us observe that the given definition of $\mathcal{L}$-congruence enable us to define the concepts of reduced product and ultraproduct of a family $((A_i, a_i, L_i))_{i \in I}$

of $\mathcal{L}$- algebras. Indeed if F is a filter on I and θ and ψ are the congruence generated by F in $\prod A_i$ and $\prod L_i$ respectively, and $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are elements of $\prod A_i$, then

$$(x_i)_{i\in I} \equiv_\theta (y_i)_{i\in I} \text{ implies } \{i \in I \colon x_i = y_i\} \in F$$

and therefore $\{i \in F\colon a_i(x_i) = a_i(y_i)\} \in F$. Thus, it follows that $(a_i(x_i))_{i\in I} \equiv_\psi (a_i(y_i))_{i\in I}$, that is $a\,[(x_i)_{i\in I}] = a\,[(y_i)_{i\in I}]$. This proves that (θ, ψ) is an $\mathcal{L}$-congruence of $(\prod A_i, \prod a_i, \prod L_i)$. We call reduced product (or ultraproduct if F is an ultrafilter) the relative quotient.

5. $\mathcal{L}$-congruences generated by equation.

The following theorem shows that the intersection of a family $((\theta_i, \psi_i))_{i\in I}$ of $\mathcal{L}$-congruences is an $\mathcal{L}$-congruence.

Theorem 5.1. The intersection $(\theta, \psi) = \bigwedge_{i\in I}(\theta_i, \psi_i) = (\bigwedge_{i\in I} \theta_i, \bigwedge_{i\in I} \psi_i)$ of a family $((\theta_i, \psi_i))_{i\in I}$ of $\mathcal{L}$- congruences of (A, a, L) is an $\mathcal{L}$-congruence of (A, a, L).

Proof. It suffices to observe that, for every $x, y \in A$, from $x \equiv_\theta y$ it follows that $x \equiv_{\theta_i} y$ and therefore $a(x) \equiv_{\psi_i} a(y)$ for every $i \in I$. Then $a(x) \equiv_\psi a(y)$ and this proves that (θ, ψ) is an $\mathcal{L}$-congruence.

From Thm. 5.1 it follows that the class of all $\mathcal{L}$-congruences of (A, a, L) forms a closure system with $(A \times A, L \times L)$ as greatest element. Furthermore if $\alpha \supseteq A \times A$ and $\beta \supseteq L \times L$ are binary relations on A and L respectively, then

$$(\overline{\alpha, \beta}) = \cap \{(\theta, \psi) : (\theta, \psi) \text{ is an } \mathcal{L} - \text{congruence of } (A, a, L)\}$$

is an $\mathcal{L}$-congruence, the $\mathcal{L}$-congruence generated bu (α, β). Since we can represent α and β as sets of equations of type $\{x = y/(x, y) \in \alpha\}$ and $\{u = v/(u, v) \in \beta\}$, then we also claim that $(\overline{\alpha, \beta})$ is the $\mathcal{L}$-congruence generated by the systems of equations α and β.

Theorem 5.2. If $\overline{\alpha}$ and $\overline{\beta}$ denote the congruences of A and L generated by α and β respectively, then, in general, it is $(\overline{\alpha,\beta}) \neq (\overline{\alpha}, \overline{\beta})$.

Proof. Let $A = B^+$ be the free semigroup with generator set B, $L = N$, where N is the real numbers set, and $l: B^+ \to L$ the lenght function. Then (A, l, L) is an $\mathcal{L}$-algebra. Let us assume $\alpha = \{(x, x^2): x \in B^+\}$ and $\beta = \{(u, u): u \in L\}$. Then $\overline{\beta} = \beta$ and $\overline{\alpha}$ is the well known congruence for which $B^+/\overline{\alpha}$ is the free band with generator set B. If $(\theta, \psi) = (\overline{\alpha, \beta})$ is the $\mathcal{L}$-congruence generated by (α, β), then, in particular, from $yx \equiv_\theta yx^2$ it follows that $l(yx) = l(yx^2)$ and therefore that $l(y) + l(x) \equiv_\psi l(y) + 2(l(x))$. This proves that $x + y \equiv_\psi x + 2y$ for every $x, y \in N$. Thus, if m and n are two elements of N and $m \geq n$ then

$$m = (2m - n) + n - m \equiv_\psi (2m - n) + 2(n - m) \equiv_\psi n.$$

This proves that $\psi = N \times N$ and therefore that $\overline{\beta} = \beta \neq \psi$.

6. Subcategories of $\mathcal{C}(\tau)$.

There are several ways for obtaining subcategories of $\mathcal{C}(\tau)$. One can put conditions on the algebras obtaining for example, the subcategory of the $\mathcal{L}$-semigroups, $\mathcal{L}$-groups, etc. In this manner we obtain the natural framework for a general treatment of the questions examined in literature.

The closure properties of these classes of algebras assure analogous closure properties for these subcategories. So, for example, the category of $\mathcal{L}$-semigroups is closed with respect to intersections of $\mathcal{L}$- subsemigroups of a given $\mathcal{L}$-semigroup, direct products and quotients.

Moreover one can put conditions on the lattices, the morphisms, etc.

In this section we examine the subcategory $\mathcal{C}(L, \tau)$ of $\mathcal{C}(\tau)$, where L is a fixed complete lattice, whose objects are $\mathcal{L}$- algebras (A, a, L) and whose morphisms are the $\mathcal{L}$-homomorphisms (h, k) such that k is the identity. In this case we call (h, k) <u>$\mathcal{L}$-homomor-</u>

phism and we denote it by h. We also denote by (A, a) the objects of $\mathcal{C}(L, \tau)$.

The hypothesis that k is the identity is assumed in order to obtain the closure of $\mathcal{C}(L, \tau)$ with respect the quotients. Indeed, if k is not the identity, then $L_a/\psi_k \neq L_a$ and this proves that the quotient $(A/\theta_h, a', L_a/\psi_k)$ is not an object of $\mathcal{C}(L, \tau)$. For the same reason we consider in $\mathcal{C}(L, \tau)$ only $\mathcal{L}$- congruences (θ, ψ) such that ψ is the equality. In this case we denote (θ, ψ) simply by θ and (θ, ψ) is named L-congruence. Now, let us prove for $\mathcal{C}(L, \tau)$ an homomorphism theorem.

Theorem 6.1. Let $h\colon (A, a) \to (B, b)$ be an L-homomorphism, then the following propositions hold:

(i) the relation $\theta_h = \{(x, y) \in A^2\colon h(x) = h(y)\}$ is a L- congruence of (A, a) and every L-congruence is of a such kind;

(ii) if h is an L-epimorphism then (B, b) is isomorphic to the quotient of (A, a) via θ_h;

(iii) if L is infinitely distributive, then the image of an L- subalgebra of (A, a) is an L-subalgebra of (B, b).

Proof. It suffices to apply the theorem 4.2, 4.3 and 4.4.

Note that the intersection of a family of L-congruences of (A, a) is an L-congruence, but that the class of L-congruences of (A, a) is not a closure system. Indeed if α is a subset of A^2 it is possible that $\{\theta\colon \theta \text{ is an } L\text{-congurence and } \alpha \leq \theta\} = \emptyset$. The following theorem gives a simple example.

Theorem 6.2. Let B^+ the free semigroup with generator set B and $l\colon B^+ \to N$ the lenght function. Then a set $\alpha \subseteq B^+ \times B^+$ of equations generates an L-congruence $\overline{\alpha}$ of (B^+, l, N) if and only if $l(x) = l(y)$ for every $(x, y) \in \alpha$. In this case $\overline{\alpha}$ coincides with the congruence of B^+ generated by α.

Proof. Let $\overline{\alpha}$ be the L-congruence generated by α, then $(x, y) \in \alpha$ implies $x \equiv_{\overline{\alpha}} y$ and therefore $l(x) = l(y)$. Conversely, let us assume that $l(x) = l(y)$ for every $(x, y) \in \alpha$, and let $\overline{\alpha}$ be the congruence of B^+ generated by α. Now if $x \equiv_{\overline{\alpha}} y$ then there exists a chain $x_o, \cdots x_n$ of elements of A with $x_0 = x$, $x_n = y$ and such that for every $i = 0, \cdots, n-1$ x_{i+1} is connected to x_i by an elementary α-transition [10], that is there exist $x, y, y_{i+1}, y_i \in A$ such that $x_{i+1} = xy_{i+1}y$; $x_i = xy_iy$ and $(y_{i+1}, y_i =\in \alpha$ or $(y_i, y_{i+1}) \in \alpha$. It follows that

$$l(x_{i+1}) = l(x) + l(y_{i+1}) + l(y)$$
$$= l(x) + l(y_i) + l(y) = l(x_i),$$

and therefore $l(x) = l(y)$.

From Thm. 6.2 it follows, for example, that there is no L-congruence containing $\alpha = \{(x, x^2) \colon x \in B^+\}$ and that there exists the L-congruence generated by $\alpha = \{(xy, yx) \colon x, yz \in B^+\}$. This L- congruence coincides with the congruence of B^+ generated by α and the quotient of (B^+, l, N) $(B^+/\overline{\alpha}, l', N)$ is the free commutative semigroup togheter with the lenght function.

7. The n-algebras.

Now we examine the category $\mathcal{C}(L_n, \tau)$ where n is a natural number and $L_n = \{1, \cdots, n\}$ is the chain with n elements. We will show that in this case there is a simple interpretation of the concepts given in this paper.

We call n-chain every chain $A_1 \supseteq \cdots \supseteq A_n$ of algebras of type τ, where $A_i \supseteq A_{i+1}$ means A_{i+1} substructure of A_i. An n-morphism h from $A_1 \supseteq \cdots \supseteq A_n$ to an n-chain $B_1 \supseteq \cdots \supseteq B_n$ is a homomorphsim from A_1 to B_1 such that, for every $i \in L_n$ and $x \in A_1$, $x \in A_i$ if and only if $h(x) \in B_i$. An n-congruence of $A_1 \supseteq \cdots \supseteq A_n$ is a congruence θ of A_1 such that every A_i is union of classes modulo θ. The n-quotient of an n-chain $A_1 \supseteq \cdots \supseteq A_n$ by an n- congruence is the n-chain $A_1/\theta \supseteq \cdots \supseteq A_n/\theta$.

Theorem 7.1. We can identify the L_n-algebras, the L_n- homomorphisms, the L_n-congruences and the quotients with the n-chains, the n-homorphisms, the n-congruences and the n-quotients respectively.

Proof. To every L_n-algebra (A, a, L_n) we can associate the n-chain $C_a^1 \supseteq \cdots \supseteq C_a^n$. Viceversa if $A_1 \supseteq \cdots \supseteq A_n$ is an n-chain let us define $a\colon A_1 \to L_n$ by setting, for every $x \in A_1$, $a(x) = \max\{i \in L_n\colon x \in A_i\}$. Then $C_a^i = A_i$. Indeed $x \in C_a^i$ implies that $a(x) \geq i$, and therefore that $A_{a(x)} \subseteq A_i$. Since $x \in A_{a(x)}$ it is also $x \in A_i$. Conversely, if $x \in A_i$ then, by definition $a(x) \geq i$, i.e., $x \in C_a^i$. This proves that all cuts of a are subalgebras of A_1, and therefore that (A_1, a, L_n) is an L_n-algebra. Then to every n-chain $A_1 \supseteq \cdots \supseteq A_n$ corresponds an L_n-algebra whose chain of cuts coincides with $A_1 \supseteq \cdots \supseteq A_n$.

Now let h be an L_n-homomorphism from (A, a, L_n) to (B, b, L_n). Then, since $a = bh$,

$$x \in C_a^i \Leftrightarrow a(x) \geq i \Leftrightarrow b(h(x)) \geq i \Leftrightarrow h(x) \in C_b^i.$$

This proves that h is an n-homomorphism from $C_a^1 \supseteq \cdots \supseteq C_a^n$ to $C_1^b \supseteq \cdots \supseteq C_b^n$. Conversely let h be such n-homomorphism, then, for every $i \in L_n$, $x \in C_a^i$ if and only if $h(x) \in C_b^i$. It follows that $a(x) \geq i$ if and only if $b(h(x)) \geq i$. By setting $i = a(x)$ we have $b(h(x)) \geq a(x)$, by setting $i \doteq b(h(x))$, we have $a(x) \geq b(h(x))$. Then $(b(h(x)) = a(x)$ for every $x \in A$ and this proves that h is a L_n-homomorphism.

Let θ be an L_n-congruence of (A, a, L_n) and let us suppose that $x \equiv_\theta y$ and $x \in C_a^i$. Then from $a(x) \geq i$ and $a(x) = a(y)$ it follows that $a(y) \geq i$, i.e. $y \in C_a^i$. This proves that θ is an n-congruence of the n-chain $C_a^1 \supseteq \cdots \supseteq C_a^n$. Conversely let θ be an n-congruence of the n-chain. Then form $x \equiv_\theta y$ and $x \in C_a^i$ it follows that $y \in C_a^i$, i.e. from $x \equiv_\theta y$ and $a(x) \geq i$ it follows that $a(y) \geq i$, for every $i \in L_n$. By setting $i = a(x)$ we obtain that $x \equiv_\theta y$ implies $a(y) \geq a(x)$. Now from $x \equiv_\theta y$ it follows also $y \equiv_\theta x$ and, by utilizing the above result, $a(x) \geq a(y)$. In conclusion $a(x) = a(y)$ and this proves that θ is an L_n-congruence. Finally, let us observe that if θ is an L_n-congruence of (A, a, L_n) and

(A', a', L_n) the relative quotient, then

$$C^i_{a'} = \{[x]_\theta : a'([x]_\theta) \geq i\}$$
$$= \{[x]_\theta : a'(x) \geq i\} = C^i_a/\theta.$$

From Thm. 7.1 we get, in particular, that if $n = 1$ then we can identify the L_1-algebras, L_1- homomorphisms, L_1-congruences and L_1-quotients with the classical concepts with the same name. This shows that all concepts given in this paper generalize the classical ones.

Theorem 7.1 enable us also to give simple characterizations of the L_n-congruences. For example if τ is the type of groups and (G, a, L_n) a L_n-group, then the L_n-congruences are just the congruences θ determined by a normal subgroup H such that $H \subseteq C^n_a$. Indeed let us suppose $H \subseteq C^n_a$. Then from $x \equiv_\theta y$ and $x \in C^i_a$ it follows that $x^{-1}y \in H \subseteq C^n_a \subseteq C^i_a$ and therefore that $y = xx^{-1}y \in C^i_a$. This proves that θ is an n-congruence of $C^1_a \supseteq \cdots \supseteq C^n_a$ and therefore an L_n-congruence of (G, a, L_n). Conversely if θ is an L_n-congruence determined by the normal subgroup H, then from $x \in H$ it follows that $x \equiv_\theta 1$. Since $1 \in C^n_a$ we have also that $x \in C^n_a$. In conclusion $H \subseteq C^n_a$.

References.

[1] Biacino, L. and Gerla, G., (1984) "Closure systems and L- subalgebras". *Information Sciences,* 32, 181-195.

[2] Di Nola, A. and Gerla, G., (1986) "Fuzzy models of a first order language", *Zeit. f. math Logik und Grund. d. Math.* 32, 331-340.

[3] Gerla, G., (1984) "Some elementary concepts of L-semigroup theory". *Ricerche di Matematica* 33, 53-62.

[4] Gerla, G., (1985) "L-subsemigroups of a free semigroup", *Rendiconti di Matematica* 5, 19-28.

[5] Gerla, G., (1985) "Pavelka's fuzzy logic and free L- subsemigroups", *Zeit. f. math. Logik und Grund. d. Math.* 31, 123-129.

[6] Gerla, G., (1986) *The category of the fuzzy models and Lowenheim-Skolem theorem* in "Mathematics of Fuzzy Systems", A. Di Nola and A. Ventre Ed., Verlag TUV Rheinland, W. Germany, 121-141.

[7] Gerla, G., (1987) "Code theory and fuzzy subsemigroups", *J. Math. Anal. Appl.* 128, 362-369.

[8] Goguen, J.A., (1967) "L-fuzzy sets", *J. Math. Anal. Appl.* 18, 145-171.

[9] Gratzer, G., (1961) *Universal Algebra*, D. Van Nostrand Comp.

[10] Howie, J.M., (1976) An introduction to Semigroup Theory. Academic Press, London.

[11] Negoita, C.V. and Ralescu, D.A., (1976) Applications to Fuzzy Sets to System Analysis, Birkauser Verlag, Basel.

[12] Rosenfeld, A., (1971) "Fuzzy Groups", *J. Math. Anal. Appl.*, 35, 512-517.

[13] Sessa, S., (1984) "On fuzzy subgroups and fuzzy ideals under triangular norms", *Fuzzy Sets and Systems*, 13, 95-100.

[14] Zadeh, L.A., (1965) "Fuzzy sets", *Inf. and Control*, 8.

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