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SINGULAR BILATERAL BOUNDARY VALUE PROBLEMS FOR DISCRETE GENERALIZED LYAPUNOV MATRIX EQUATIONS

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ABSTRACT

Existence and uniqueness conditions for solving singular bilateral initial and two-point boundary value problemes for discrete generalized Lyapunov matrix equations and explicit expressions of solutions are given.

1. Introduction.

Implicit vectorial systems of the form $F(x_{k+1}, x_k, k) = 0$ arise in a variety of economic, systems theory and signal processing problems [2], [3], [7], [8]. The theory of the linear case $A_k x_k + B_k x_k = f_k$ is reasonably complete on a finite interval $\Gamma_N = [0, 1, ..., N]$, where N is a positive integer. The nonlinear case has been studied in [4] and on the infinite time interval $\Gamma = [0, 1, ...]$. Throughout, the set of $m \times n$ matrices over a field $\mathcal{K}$ will be denoted by $\mathcal{K}_{m,n}$. Moreover, a g-inverse of a $\mathcal{K}_{m,n}$ will be denoted by A^- and understood as a matrix for which $AA^-A = A$ (see [2] for details).

In this paper we are interested in the study of singular bilateral boundary value problems for the discrete generalized Lyapunov matrix equation

$$B_k X_{k+1} A_k - X_k = C_k, \quad k \in \Gamma_N \tag{1.1}$$

where $A_k \in \mathcal{K}_{m,n}$, $B_k \in \mathcal{K}_{m,n}$, $C_k \in \mathcal{K}_{m,n}$, $X_k \in \mathcal{K}_{m,n}$. The discrete equation (1.1) is interesting from a practical point of view because it results from a discretization of the corresponding continuous equation. It has also a theoretical interest as it has been proved in [12], where the asymptotic stability of a discrete periodic linear system is related to the periodic, positive definite, symmetrical solution set of a periodic discrete Lyapunov equation. In section 2 the following problems will be studied

$$\begin{cases} B_k X_{k+1} A_k - X_k = C_k \\ E X_o F = G \\ k \in \Gamma_N \end{cases} \tag{2.1}$$

$$\begin{cases} B_k X_{k+1} A_k - X_k = C_k \\ P X_o Q = G \\ R X_j S = L \\ j, k \in \Gamma_N \end{cases} \tag{3.1}$$

and

$$\begin{cases} B_k X_{k+1} A_k - X_k = C_k \\ E X_o F - G X_j H = L \\ j, k \in \Gamma_N \end{cases} \tag{4.1}$$

where $E, P \in \mathcal{K}_{m,m}$, $F, Q, S \in \mathcal{K}_{n,n}$ and $G, L \in \mathcal{K}_{m,n}$.

Initial conditions such as the one of (2.1) and boundary value conditions such us the ones of (3.1) and (4.1) can be regarded as singular bilateral conditions because they can not be reduced to unilateral ones and they are more general that the ones considered in [2], [3]. These conditions are related to the resolution problem of bilateral matrix equations of the types $AXB = C$ and $AXB - CXD = E$ which arise in the eigenstructure method for pole assignment by output feedback [6], the theory of estimating covariance components in a covariance components model, [14], etc.

If A belongs to $\mathcal{K}_{m,n}$ we denote by A^T the conjugate transposed matrix of A and if $m = n$, the set of eigenvalues of A will be denoted by $\sigma(A)$. The expression $A > 0$ means that the matrix A is positive definite.

2. Singular bilateral initial and boundary value problems.

We begin this section with a result that establishes a necessary and sufficient condition for the consistence of the problem (2.1) so as an explicit expression for solutions. For convenience we consider the associated linear discrete systems

$$X_{k+1} = A_k X_k \tag{1.2}$$

$$Y_{k+1} = B_k^T Y_k \tag{2.2}$$

and let Φ and Ω the transition state matrix of (1.2) and (2.2) respectively and let $\Phi_{k,j}$ and $\Omega_{k,j}$ their action on the pair (k,j) $\Gamma_N \times \Gamma_N$. We recall that for instance $\Phi_{k,j} = A_{k-1}A_{k-2}\cdots A_j$, if $k > j$ and $\Phi_{j,j} = I$.

Theorem 1. Let us consider the matrices

$$A = E\Omega_{N,o}^T; \quad B = \Phi_{N,o}F; \quad C = -E\sum_{j=o}^{N-1}\Omega_{j,o}^T C_j \quad \Phi_{j,o}F + G \tag{3.2}$$

then the problem (2.1) is consistent if and only if for some matrices A^- and B^- it is satisfied that

$$AA^-C = C \quad \text{and} \quad CBB^- = C \tag{4.2}$$

In this case, the general solution of (2.1) is given by the expression

$$X_j = \Omega_{N,j}^T X_n \Phi_{N,j} - \sum_{p=1}^{N-1}\Omega_{p,j}^T C_p \Phi_{p,j}, \quad j \in \Gamma_{N-1} \tag{5.2}$$

where

$$X_N = A^-CB^- + U + A^-AUBB^- \tag{6.2}$$

with arbitrary $U \in \mathcal{K}_{m,n}$.

Proof. By backward sustitution it is easy to show that the sequence defined by $(5,2)$ with X_N given by (6.2) satisfies the equation (1.1) in Γ_N. The bilateral initial condition in (2.1) is equivalent to the existence of a matrix X_N the satisfies the equation

$$E(\Omega_{N,o}^T X \Phi_{N,o} - \sum_{p=1}^{N-1} \Omega_{p,o}^T C_p \Phi_{p,o})F = G$$

From (3.2) it is equivalent to the equation $AXB = C$. Now, the result is a consequence of lemma 1 of [1].

Corollary 1. Let us consider the problem (2.1) with $\mathcal{K}$ is the field of real numbers and $m = n$. Suppose thta $G > 0$, $-C_k > 0$, $E = F^T$ and $A_k = B_k^T$ for every k in Γ_N. If the hypothesis (4.2) is satisfied, then for any positive defined matrix $U \in R_{n,n}$ the sequence (5.2) defines a positive definite solution of the corresponding problem (2.1).

Proof. From the hypothesis $A_k = B_k^T$ for $k \in \Gamma_N$, it follows that $\Phi_{N,k} = \Omega_{N,k}$. From the expressions $(3.2), (5.2)$ and theorem 1 the result is concluded.

The following result solves the problem (3.1).

Theorem 2. Let us consider the matrices

$$A = P\Omega_{N,o}^T; \quad B = \Phi_{N,o}Q; \quad C = G - P\sum_{p=o}^{N-1} \Omega_{p,o}^T C_{p,o} Q$$

$$D = R\Omega_{N,j}^T; \quad E = \Phi_{N,j}S; \quad F = L - K\sum_{p=j}^{N-1} \Omega_{p,j}^T C_p \Phi_{p,j} J \tag{7.2}$$

Then the problem (3.1) is consistent if and only if

$$A^T A(A^T A + D^T D)^- D^T F E^T (BB^T + EE^T)^- BB^-$$

$$= D^T D(A^T A + D^T D)^- A^T C B^T (BB^T + EE^T)^- EE^T \tag{8.2}$$

In this case the general solution (3.1) is given by (5.2) where X_N is given by the expression

$$X_N = (A^TA + D^TD)^-(A^TCB^T + Y + Z + D^TFE^T)(BB^T + EE^T)^- + U$$

$$-(A^TA + D^TD)^-(A^TA + D^TD)U(BB^T + EE^T)(BB^T + EE^T) \tag{9.2}$$

where U is an arbitrary matrix in $\mathcal{K}_{m,n}$. The matrices Y and Z are arbitrary solutions of

$$Y(BB^T + EE^T)BB^T = A^TCB^T(BB^T + EE^T)^-EE^T$$

$$Z(BB^T + EE^T)^-EE^T = D^TFE^T(BB^T + EE^T)^-BB^T \tag{10.2}$$

Proof. Fron the lemma 1 the consistency of the problem (3.1) is equivalent to the existence of a solution $X = X_N$ of the algebraic system

$$P(\Omega_{N,o}^T X\Phi_{N,o} - \sum_{p=1}^{N-1} \Omega_{p,o}^T C_p \Phi_{p,o})Q = G$$

$$R(\Omega_{N,j}^T X\Phi_{N,j} - \sum_{p=j}^{N-1} \Omega_{p,j}^T C_p \Phi_{p,j})S = L$$

Taking into account the expressions contained in (7.2) it is equivalent to the consistency of the system

$$\begin{cases} AXB = C \\ DXE = F \end{cases}$$

Now the result is a consequence of [10].

The following results is concerned with the resolution of the two-point boundary value problem (4.1) . Before of its statement, we note that given square matrices A, B, C and D in $\mathcal{K}_{n,n}$, there are at most n values of λ such tht the determinants $|C - \lambda A|$ and $|B + \lambda D|$ are annihilated simultaneously, bein λ a complex number. This observation is related with the hypothesis of the following theorem.

Theorem 3. Let us consider the matrices $A = E\Omega_{N,o}^T$; $B = \Phi_{N,o}F$; $C = -G\Omega_{N,j}^T$; $D = \Phi_{N,j}H$; $R = E\sum_{p,o}^{N-1}\Omega_{p,o}^T C_p \Phi_{p,o}F - G\sum_{p=j}^{N-1}\Omega_{p,j}^T C_p\Phi_{p,j}H + L$.

(i) The problem (4.1) is solvable if and only if the equation $AXB+CXD = R$ is consistent. In this case the general solution of (4.1) is given by (5.2) being X_N a solution of $AXB + CXD = R$.

(ii) The problem (4.1) is solvable if and only if for any complex λ such that

$$|C - \lambda A| \neq 0 \quad \text{and} \quad |B + \lambda D| \neq 0 \tag{11.2}$$

the block matrices

$$\begin{pmatrix} (C-\lambda A)^{-1}A & 0 \\ 0 & -D(B+\lambda D)^{-1} \end{pmatrix} ; \begin{pmatrix} (C-\lambda A)^{-1}A & (C-\lambda A)^{-1}R(B+\lambda D)^{-1} \\ 0 & -D(B+\lambda D)^{-1} \end{pmatrix}$$

are similar.

(iii) If for some λ such that the condition (11.2) is satisfied, the spectrums $\sigma((C-\lambda A)^{-1}A)$ and $\sigma(-D(B + \lambda D)^{-1})$ have empty intersection, then the problem (4.1) has only one solution.

(iv) Under the hypothesis of (iii) and if $p(z) = \sum_{k=o}^{n} p_k z^k$ is the characteristic polynomial of $(C - \lambda A)^{-1}A$, then the only solution of (4.1) is given by (5.2) where X_N is given by the expression

$$X_N = (\sum_{k=o}^{n} p_k((C-\lambda D)^{-1}A)^k)^{-1}$$

$$(\sum_{k=1}^{n}\sum_{r=1}^{k} p_r((C-\lambda D)^{-1}A)^{r-1}((C-\lambda A)^{-1}R(B+\lambda D)^{-1}(-D(B+\lambda D)^{-1})^{k-r}$$

Proof. (i) From the proof of the lemma 1 it is clear that the sequence given by (5.2) satisfies the equation (1.1) . This sequence defines a solution of the boundary value problem (4.1) if and only if the following condition is verified

$$E(\Omega_{N,o}^T X_N \Phi_{N,o} - \sum_{p=o}^{N-1}\Omega_{p,o}^T C_p \Phi_{p,o})F$$

$$-G(\Omega_{N,k}^T X_N \Phi_{N,j} - \sum_{p=j}^{N-1} \Omega_{p,j}^T C_p \Phi_{p,j}) H = L$$

That is, if $X = X_N$ is a solution of the equation $AXB + CXD = R$.

(ii) From the hypothesis (11.2) and [13], the equation $AXB + CXD = R$ is consistent if and only if the equation

$$(C - \lambda A)^{-1} AX + XD(B + \lambda D)^{-1} = (C - \lambda A)^{-1} R(B + \lambda D)^{-1} \qquad (12.2)$$

is consistent. Moreover, from [9] the result is concluded.

(iii) From the hypothesis and the Rosenblun theorem [8] , the equation (12.2) has only one solution. From (i) and (ii) the result is proved.

(iv) The result is a consequence of (iii) and [5] .

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