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EVALUATIONS OF FUZZY SETS BASED ON ORDERINGS AND MEASURES

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ABSTRACT

Total orderings on the range of fuzzy sets can serve as choice criteria for fuzzy sets, a wide class of orderings based on functions is proposed (section 2). Decomposable measures are taken to measure the items on which the fuzzy sets are given (section 3). Combining the two levels of measurement by means of the integral introduced by the second author we obtain evaluations of fuzzy sets as functionals with appropriate properties, the concepts of energy and fuzziness are included (section 4).

1. Introduction.

It has been stressed by several authors the role of fuzzy set theory in the analysis of impreciseness that occurs in the descriptions of concepts, systems, see e.g. [7, 8] and references therein. The search for a unique context fitting out either fuzziness, i.e. the impreciseness due to the occurrene of an event, that can arise simultaneously is one of the issues of great interest, see e.g. [9, 10, 11, 12].

If a fuzzy set is considered as a "global description" of a fixed universal set, it is natural to try to evaluate such a description and, in particular, to answer the question how precise and significant is the description, see e.g. [7, 14]. Special evaluations of fuzzy sets were studied by De Luca and Termini [1, 2] by means of their concepts of energy and entropy of a fuzzy set in order to calculate the amounts of membership and fuzziness of a fuzzy set, respectively.

More general, an evaluation of fuzzy sets can be given in two steps:

First, certain characteristic of a fuzzy set can be described by an order relation for the values of the fuzzy set.

Second, this characteristic can be measured by an appropriate integral based on some measure.

Knopfmacher [5] applied this principle to define a measure of fuzziness based on a probability. Later on Di Nola and Ventre [3, 4] and Weber [11, 12] used this principle dealing with some generalizations at the first and at the second step.

The aim of this paper is to deal with a unified presentation of these ideas.

2. Orderings as choice criteria

Let $\mathcal{L}'(\Omega)$ denote the set of all fuzzy subsets $\varphi : \Omega \to [0,1]$ of a universe Ω.

Let us assume a partial ordering $\leq_0$ on the range $[0,1]$ of the fuzzy sets. A partial ordering on $\mathcal{L}'(\Omega)$ will be induced by

$$\varphi \leq_0 \psi \quad iff \quad \varphi(\omega) \leq_0 \psi(\omega) \quad \text{for every} \quad \omega \in \Omega, \tag{2.1}$$

where we use the same symbol for the ordering on $[0,1]$ and on $\mathcal{L}'(\Omega)$.

If $([0,1], \leq_0)$ is a lattice, being $\wedge_0$ and $\vee_0$ meet and join respectively, then $(\mathcal{L}'(\Omega), \leq_0)$ is a lattice too. Particularly, if $\leq_0$ is a total ordering on $[0,1]$, then $([0,1], \leq_0)$ is a lattice.

The natural ordering $\leq$ on $[0,1]$ is total with $\wedge = min$, $\vee = max$ and induces the lattice structure on $\mathcal{L}'(\Omega)$ originally proposed by Zadeh. Negoita [7] interpreted $\varphi \wedge \psi$ as synthesis of φ and ψ and $\leq$ as a choice criterion.

We will use these interpretations for any meet $\wedge_0$ and ordering $\leq_0$ on $\mathcal{L}'(\Omega)$ based on a total ordering $\leq_0$ on $[0,1]$. In order to obtain not only a qualitative criteron $\leq_0$ but a quantitative one, we generalize the construction from $[3,4]$ in the following.

Definition 1. Given any function $K:[0,1]\to[0,1]$, we set

$$x \leq_K y \quad \text{iff} \begin{cases} K(x) < K(y) \\ \quad \text{or} \\ K(x) = K(y), \quad x \leq y \end{cases} \tag{2.2}$$

It can be seen that every relation $\leq_K$ from (2.2) is a total ordering on $[0,1]$ and that

$$x \leq_K y \quad \text{implies} \quad K(x) \leq K(y), \tag{2.3}$$

i.e. K is an order morphism: $([0,1],\leq_K) \to ([0,1],\leq)$.

Example 1. Every increasing function $E:[0,1]\to[0,1]$ with $E(0)=0$ and $E(1)=1$ yields $\leq_E = \leq$. We will call E an <u>energy function</u>, see section 4.

Note that also other functions K lead to the natural ordering, e.g. a "three- decision-function" as

$$K(x) := \begin{cases} 0 & \text{if} \quad x < \epsilon \\ 0.5 & \text{if} \quad x \in [\epsilon, 1-\epsilon] \quad \epsilon \in (0, 0.5]. \\ 1 & \text{if} \quad x > 1-\epsilon \end{cases} \tag{2.4}$$

Example 2. Let F be a <u>fuzziness function</u> in the sense of [12], i.e. a function F : $[0,1]\to[0,1]$, increasing in $[0,u_{max}]$, decreasing in $[u_{max},1]$ such that $F(0)=0$ and $F(n(x))=F(x)$, where $n:[0,1]\to[0,1]$ is decreasing and continuous with $n(0)=1$ and $n(1)=0$ and u_{max} is the unique fixpoint of (the negation) n. The derived (total) ordering $\leq_F$ is not the natural one, particularly,

$$0 \leq_F 1 \leq_F x \leq_F u_{max} \quad \text{for all} \quad x \neq 0 \tag{2.5}$$

Let us denote by $\leq_S$ the "<u>sharpening</u>" (partial) <u>ordering</u> in the sense of [1] with the modification of [4, 12],

$$x \leq_S y \quad \text{iff} \begin{cases} x \leq y & \text{if} \quad x, y \leq u_{max} \\ x \geq y & \text{if} \quad x, y \geq u_{max} \end{cases} \tag{2.6}$$

Then we see that, for every fuzziness function,

$$x \leq_S y \quad \text{implies} \quad x \leq_F y \tag{2.7}$$

and therefore by (2.3) that F is an order morphism:

$$([0,1], \leq_S) \to ([0,1], \leq)$$

Using (2.1), $\varphi \leq_S \psi$ will be interpreted as "φ is sharper than ψ" and $\varphi \dot{\leq}_F \psi$ as "φ is less fuzzy than ψ" with respect to F. The latter notion is a reasonable extension of the former one, but it depends on F which shall reflect a concrete (quantified) meaning of "fuzziness".

3. Decomposable measures

We will evaluate some characteristic of a fuzzy subset φ of Ω by an ordering $\leq_K$ from section 2, taking into consideration the "importance of the items" $\omega \in \Omega$, for which the values $\varphi(\omega)$ are given. Following [12] this "importance" will be measured by decompsoable measures as introduced in [11] . In the present section we will recall briefly from [11] what we need in section 4.

Definition 2. A function $m : \mathcal{A} \to [0, M]$, $M \in (0, \infty]$, on a σ-algebra $\mathcal{A}$ over Ω with the boundary conditions $m(\emptyset) = 0$ and $m(\Omega) = M$ will be called a $\sigma - \perp -$ decomposable measure iff

$$m(A \cup B) = m(A) \perp m(B), \tag{3.1}$$

$$A_n \uparrow A \quad \text{implies} \quad m(A_n) \uparrow m(A), \tag{3.2}$$

where $\perp$ is a t-conorm on $[0, M]$.

Recall that a binary operation $\perp$ on $[0, M]$ is a t-conorm iff $\perp$ is non-decreasing in each argument, commutative, associative and has 0 as unit.

We restrict ourselves to the Archimedean t-conorms $\perp$ on $[0, M]$, i.e. to those binary operations $\perp$ which are characterized by the representation, [6],

$$a \perp b = g^{(-1)}(g(a) + g(b)) \tag{3.3}$$

by means of an increasing, continuous function $g : [0, M] \to [0, \infty]$ with boundary condition $g(0) = 0$ and pseudo- inverse $g^{(-1)}$ given by

$$g^{(-1}(x) := g^{-1}(min(x, g(M))). \tag{3.4}$$

The function g in (3.3) is called (additive) generator of $\perp$ and is unique up to a constant positive factor.

Furthermore, within the class of Archimedean t-conorms, the strict ones are characterized by the boundary condition.

$$g(M) = \infty. \tag{3.5}$$

From (3.1), (3,2) and (3.3) there follows the classification into the three following cases, [11].

(S) : $\perp$ is strict. Then $g \circ m$ is an infinite σ -additive measure on $\mathcal{A}$.

(NSA) : $\perp$ is non- strict and $g \circ m$ is a finite σ -additive measure on $\mathcal{A}$.

(NSP) : $\perp$ is non- strict but $g \circ m$ is pseudo- σ -additive in the sense that

$$(g \circ m)(\cup A_j) = g(M) < \sum_i (g \circ m)(A_j) \tag{3.6}$$

is possible.

More details can be found in [11, 12].

4. Evaluations of global properties

Now we fix a $\sigma-$ $\perp$-decomposable measure m on $(\Omega, \mathcal{A})$ with respect to an Archimedean t-conorm $\perp$ on $[0, M]$. Let $\mathcal{L}(\Omega)$ denote the subset of $\mathcal{L}'(\Omega)$ of all measurable fuzzy subsets of Ω, briefly denoted by $\mathcal{L}$.

The last concept we need is that of an integral as a functional $\mathcal{J} : \mathcal{L} \to [0, M]$. Let us recall from [11, 12] briefly the following.

Definition 3 A measure of fuzzy sets is defined by the integral

$$\mathcal{J}(\varphi) := \int_{\Omega} \varphi \perp m := \begin{cases} g^{-1}(\int_{\Omega} \varphi d(g \circ m)) & \text{for (S) or (NSA)} \\ g^{(-1)}(\sum_{i=1}^{\infty} \int_{\Omega_i} \varphi d(g \circ m)) & \text{for (NSP)} \end{cases} \tag{4.1}$$

where in the (NSP) case we need that Ω is m-achievable in the sense that there exists $\Omega = \cup_{i=1}^{\infty} \Omega_i$ with $m(\Omega_i) < M$. The classification from section 3 ensures that we can form the Lebesgue-integrals $\int_{\Omega}$ and $\int_{\Omega_i}$. The situation (3.6) can not occur in the subsets Ω_i. The definition is independent of the choices of the generator g and the sequence (Ω_i) .More details can be found in [11], where $M = 1$, but the properties can be modified for the general setting $M \in (0, \infty]$.

Theorem. Let $K : [0,1] \to [0,1]$ be some measurable function and $\leq_K$ the derived total ordering on $[0,1]$ from (2.2). Let $m : \mathcal{A} \to [0, M]$ be some $\sigma - \perp$ -decomposable measure with respect to an Archimedean t- conorm $\perp$ on $[0, M]$ and $\mathcal{J} : \mathcal{L} \to [0, M]$ the derived integral from (4.1). Then

$$\mathcal{K}(\varphi) := \mathcal{J}(K \circ \varphi) \tag{4.2}$$

gives a functional $\mathcal{K} : \mathcal{L} \to [0, M]$ with the following properties:

$$\text{i) } \varphi \leq_K \psi \text{ implies} \mathcal{K}\varphi \leq \mathcal{K}\psi, \tag{4.3}$$

$$\text{ii) } \varphi_n \uparrow_{(\leq_K)} \varphi \text{ implies } \mathcal{K}\varphi_n \uparrow_{(\leq)} \mathcal{K}\varphi. \tag{4.4}$$

$$\text{iii) } \mathcal{K}(\varphi \wedge_K \psi) \perp \mathcal{K}(\varphi \vee_k \psi) = \mathcal{K}\varphi \perp \mathcal{K}\psi, \tag{4.5}$$

Proof. (i) and (ii): follow from (2.1), (2.3) and [11] theorem 4.2 (ii) and (v) respectively. (iii): For cases (S) and (NSA) we obtain (4.5) directly:

$$\mathcal{K}(\varphi \wedge_K \psi) \perp \mathcal{K}(\varphi \vee_K \psi) = g^{-1}(\int K(\varphi \wedge_K \psi) + \int K(\varphi \vee_K \psi))$$

$$= g^{-1}(\int_{\{\varphi \leq_K \psi\}} K\varphi + \int_{\{\varphi >_K \psi\}} K\psi + \int_{\varphi \leq_K \psi\}} K\psi + \int_{\{\varphi >_K \psi\}} K\varphi$$

$$= g^{-1}(\int K\varphi + \int K\psi) = \mathcal{K}\varphi \perp \mathcal{K}\psi,$$

where we used the abbreviation

$$\int K\varphi \quad \text{for} \quad \int_{\Omega}(K \circ \varphi)d(g \circ m).$$

For the case (NSP) we obtain by the same splitting as above:

$$\sum_i \int_{\Omega_i} K(\varphi \wedge_K \psi) + \sum_i \int_{\Omega_i} K(\varphi \vee_K \psi) = \sum_i \int_{\Omega_i} K\varphi + \sum_i \int_{\Omega_i} K\psi. \tag{4.6}$$

In order to complete the proof of (iii) we have to distinguish two cases:

a) If $\sum_i \int_{\Omega_i} K(\varphi \vee_K \psi) < g(M)$ then, by (2.3), the other three terms in (4.6) are also less than $g(M)$. Applying the pseudo-inverse $g^{(-1)}$ to (4.6) we obtain (4.5).

b) If $\sum_i \int_{\Omega_i} K(\varphi \vee_K \psi) \geq g(M)$ then the left side in (4.5) attains the maximum value M. Applying $g^{(-1)}$ to (4.6) we obtain therefore that

$$M = g^{(-1)}(\sum_i \int_{\Omega_i} K\varphi + \sum_i \int_{\Omega_i} K\psi). \tag{4.7}$$

If both terms at the right side of (4.7) are less than $g(M)$ then both right sides of (4.5) and of (4.7) coincide. If one term (or both) at the right side of (4.7) is greater than or equal to $g(M)$ then this term can be replaced by $g(M)$ without changing (4.7). This completes the proof.

Let us illustrate the theorem looking at the examples from section 2.

Example 1: Applying the theorem to a measurable energy function $K = E$, properties (i), (ii) and (iii) hold for $\mathcal{K} =: \mathcal{T}$ with respect to the natural lattice $(\mathcal{L}, \leq, \vee, \wedge)$. Furthermore,

$$\text{iv) } \varphi = 0 \text{ a.e. iff } \mathcal{T}\varphi = 0, \tag{4.8}$$

$$\text{v) } \varphi = 1 \text{ a.e. implies } \mathcal{T}\varphi = M. \tag{4.9}$$

We call $\mathcal{T}$ an energy measure derived from E, compare with [2].

The energy $\mathcal{T}$ derived from $E = id$ is the measure of fuzzy sets from [12] , which in the probability case $m = P$ was already proposed by Zadeh [13].

Example 2: Applying the theorem to a measurable fuzziness function $K = F$, properties (i), (ii) and (iii) hold for $\mathcal{K} =: \mathcal{F}$ with respect to the lattice $(\mathcal{L}, \leq_F, \wedge_F, \vee_F)$. Furthermore,

$$\text{iv) } \varphi = 1_A \text{ a.e. iff } \mathcal{F}\varphi = 0, \tag{4.10}$$

$$\text{v) } \varphi = u_{max} \text{ a.e. implies } \mathcal{F}\varphi = M. \tag{4.11}$$

In accordance with [12] we call $\mathcal{F}$ a fuzziness measure derived from F.

We note that the extension property (2.7) gives as a corollary the following two properties corresponding to $\leq_s$:

$$\text{i)* } \varphi \leq_S \psi \text{ implies } \mathcal{F}\varphi \leq \mathcal{F}\psi, \tag{4.12}$$

$$\text{ii)* } \varphi_n \uparrow_{(\leq_S)} \psi \text{ implies } \mathcal{F}\varphi_n \uparrow_{(\leq)} \mathcal{F}\varphi. \tag{4.13}$$

But note that $(\mathcal{L}, \leq_S)$ is not a lattice, particularly a meet $\wedge_S$ does not exist in general. Therefore a property analogous to (iii) has no sense in $(\mathcal{L}, \leq_S)$.

In the proof of the theorem we needed that $\leq_K$ is a total ordering.

5. Conclusion.

The functional $\mathcal{K} : \mathcal{L} \to [0, M]$ from the theorem is an order morphism (i), a $\perp$-valuation in the sense of (iii) and continuous from above (ii) with respect to the lattice structure in $\mathcal{L}$ derived from the total ordering $\leq_K$ on $[0,1]$. This ordering shall describe some property/criterion for the values of fuzzy sets. The functional $\mathcal{K}$ measures this property for fuzzy sets and will be called evaluation of fuzzy sets.

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