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CONVEX TRANSFORMATIONS WITH BANACH LATTICE RANGE

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ABSTRACT

A closed epigraph theorem for Jensen-convex mappings with values in Banach lattices with a strong unit is established. This allows one to reduce the examination of continuity of vector valued transformations to the case of convex real functionals. In particular, it is shown that a weakly continous Jensen-convex mapping is continuous. A number of corollaries follow; among them -a characterization of continuous vector-valued convex transformations is given that answers a question raised by Ih-Ching Hsu.

It is known that a weakly continuous additive operator is bounded. This fact results immediately from the classical closed graph theorem of Banach. What about convex transformations? In many instances regularity properties of additive mappings are shared by convex functionals (see e.g. M. Kuczma [6]).

Some endeavours in this direction were also made with regard to convex transformations with values in vector spaces (see Ih-Ching Hsu and Robert G. Kuller [5] and Ih-Ching Hsu [4]). In the latter paper the author deals with weak and strong convexity as well as with some integral representation theorems. However, he confines his attention to the usual convexity although midpoint convexity (convexity in the sense of Jensen) is mentioned. Motivated by some typical argumentation frequently used in the theory of vector measures, for instance, we have tried to reduce the examination of vector-valued convex transformations to that of convex functionals.

Let us fix some notation and terminology. A real linear space is called a vector lattice (or linear lattice, or Riesz space) provided it is equiped with a lattice structure in which vector translations and scalar multiplications by positive reals are isotone in the sense tha they preserve the partial order (see e.g. Birkhoff [2]). A real Banach space $(Y, \|\cdot\|)$ is termed a Banach lattice whenever Y is a vector lattice with a partial order $\preceq$ such that

$$(1) \qquad |u| \preceq |v| \quad \text{implies} \quad \|u\| \leq \|v\|, \quad u, v \in Y,$$

where $|u| := sup\{u, -u\}, \quad u \in Y.$

If $(Y, \|\cdot\|, \preceq)$ is a Banach lattice then a positive element $e \in Y$ is called a strong unit if and only if for every $u \in Y$ there exists an $n \in N$ such that $u \prec ne$. The existence of a strong unit in a given Banach lattice $(Y, \|\cdot\|, \preceq)$ is equivalent to the order-boundedness of the unit ball in Y. In particular, that is the case where metric boundedness is equivalent to order-boundedness (see Birkhoff [2]).

A vector lattice is called boundedly complete iff each nonempty set A that has an upper bound has a least upper bound, sup A. (This forces a nonempty set A having a lower bound to possess a greatest lower bound, inf A).

Finally, let us recall that a topological space X is called a Baire space provided that each nonempty open subset of X is of the second Baire category (i.e. cannot be represented as a countable union of nowhere dense sets).

Our first result yields a "convex analogue" of the celebrated closed graph theorem of Banach.

Theorem 1. (closed epigraph theorem). Let X be a real linear topological Baire space, $(Y, \|\cdot\|, \preceq)$ -a Banach lattice with a strong unit, and D- a nonempty open and convex subset of X. If a map $f : D \to Y$ satisfies inequality

(2) $$f\left(\frac{x+y}{2}\right) \preceq \frac{f(x)+f(y)}{2}, \quad x, y \in D,$$

and if the set

(3) $$\{(x, y) \in D \times Y : f(x) \preceq y\}$$

is closed in $D \times Y$, then f is continuous.

In what follows, a map f fulfilling inequality (2) will be called Jensen- convex whereas set (3) will be termed the epigraph of f and denoted by epi f.

Proof of Theorem 1. Inequality (2) and a straightforward induction imply the relationship

$$f\left(\sum_{i=1}^{2^n} \frac{1}{2^n} x_i\right) \preceq \frac{1}{2^n} \sum_{i=1}^{2^n} f(x_i)$$

valid for any choice of points $x_i \in D$, $i \in \{1, ..., n\}$, and any $n \in N$. Setting here $x_1 = ... = x_k = x$ and $x_{k+1} = ... = x_{2^n} = y$ we get inequality

(4) $$f\left(\frac{k}{2^n} x + \left(1 - \frac{k}{2^n}\right) y\right) \preceq \frac{k}{2^n} f(x) + \left(1 - \frac{k}{2^n}\right) f(y)$$

satisfied for any $x, y \in D$, $k \in \{1, ..., 2^n\}$, and $n \in N$.

Fix arbitrarily a point $x_o \in D$ and put $\tilde{D} := D - x_o$, $g(x) := f(x + x_o) - f(x_o)$, $x \in \tilde{D}$. Then the set $\tilde{D}$ is open and convex, $0 \in \tilde{D}$, and $g(0) = 0$. Moreover, on account of (4) applied for $k = 1$,

(5) $$g\left(\frac{1}{2^n} x + \left(1 - \frac{1}{2^n}\right) y\right) \preceq \frac{1}{2^n} g(x) + \left(1 - \frac{1}{2^n}\right) g(y), \quad x, y \in \tilde{D}, \quad n \in N.$$

and, in particular,

(6) $$g\left(\frac{1}{2^n} x\right) \preceq \frac{1}{2^n} g(x), \quad x \in \tilde{D}, \quad n \in N.$$

Finally,

$$\text{epi } g = \text{epi } f - (x_o, f(x_o)),$$

which proves that the epigraph of g is closed in $\tilde{D} \times Y$ because, by assumption, epi f is closed in $D \times Y$. Let e be a strong unit in Y and let

$$L := \{x \in \tilde{D} : g(x) \preceq e\}.$$

Observe that $\tilde{D} \subset \bigcup_{n \in N} (2^n L)$. Indeed, fix an $x \in \tilde{D}$; then $g(x) \preceq 2^{n_o} e$ for some $n_o \in N$ and, by (6),

$$g\left(\frac{1}{2^{n_o}} x\right) \preceq \frac{1}{2^{n_o}} g(x) \preceq e$$

i.e. $\frac{1}{2^{n_o}} x \in L$ whence $x \in 2^{n_o} L \subset \bigcup_{n \in N} (2^n L)$. Therefore, since $\tilde{D}$, as a nonempty and open subset of a Baire space X, is of the second Baire category, one has

$$\text{int } cl(2^n L) \neq \emptyset$$

for some $n \in N$. Consequently,

$$U := \text{int } clL \neq \emptyset \tag{7}$$

Observe that L is closed in $\tilde{D}$. To see this, fix a point $a \in \tilde{D} \setminus L$. Then the pair (a, e) belongs to the open set $(\tilde{D} \times Y) \setminus \text{epi } g$. Hence, there exist neighbourhoods $U_a \subset \tilde{D}$ and $V_e \subset Y$ of points a and e , respectively, such that

$$U_a \times V_e \subset (\tilde{D} \times Y) \setminus \text{epi } g$$

In particular, $U_a \times \{e\} \subset (D \times Y) \setminus \text{epi } g$ which means that for every $x \in U_a$ we have $x \in \tilde{D} \setminus L$. Thus $\tilde{D} \setminus L$ is open, i.e. L is closed in $\tilde{D}$, as claimed.

Now, by virtue of (7),

$$\emptyset \neq U \cap L = U \cap \tilde{D} \cap \text{cl } L = U \cap \tilde{D}$$

Therefore, $U \cap L$ is a nonempty open set contained in L which means that

$$V := \text{int } L \neq \emptyset$$

Fix a $v_o \in V$ and choose $n \in N$ large enough to have $u_o := \frac{1}{1-2^n} v_o$ in $\tilde{D}$ (which is possible because $\tilde{D}$ yields a neighbourhood of zero). Plainly, the set

$$V_o := \frac{1}{2^n}(V - v_o)$$

is a neighbourhood of zero. Let $x \in V_o$; then, for some $v \in V$, one has

$$x = \frac{1}{2^n}(v - v_o) = \frac{1}{2^n}v + \left(1 - \frac{1}{2^n}\right) u_o$$

whence $x \in \tilde{D}$ by convexity of $\tilde{D}$ and, in view of (5),

$$g(x) \preceq \frac{1}{2^n} g(v) + \left(1 - \frac{1}{2^n}\right) g(u_o) \preceq \frac{1}{2^n} e + \left(1 - \frac{1}{2^n}\right) g(u_o) =: c \preceq |c|$$

Thus

$$(8) \qquad V_o \subset \tilde{D} \quad \text{and} \quad g\,|_{V_o} \preceq |c|$$

Fix arbitrarily an $\epsilon > 0$ and take an $n_o \in N$ such that $\frac{1}{2^{n_o}} < \frac{\epsilon}{1 + \|c\|}$. Obviously, $U_o := \frac{1}{2^{n_o}}(V_o \cap (-V_o))$ yields a symmetric neighbourhood of zero. For $x \in U_o$ we have $2^{n_o}x \in V_o$ whence, by (6) and (8),

$$g(x) = g\left(\frac{1}{2^{n_o}}(2^{n_o}x)\right) \preceq \frac{1}{2^{n_o}} g(2^{n_o}x) \preceq \frac{1}{2^{n_o}}|c|.$$

On the other hand, inequality (5) applied for $n = 1$ and $y = -x$ yields

$$0 = g(0) = g\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) \preceq \frac{1}{2}g(x) + \frac{1}{2}g(-x)$$

i.e.

$$-g(x) \preceq g(-x) \preceq \frac{1}{2^{n_o}}|c| \quad \text{for} \quad x \in U_o = -U_o.$$

Consequently,

$$|g(x)| = \sup\{g(x), -g(x)\} \preceq \frac{1}{2^{n_o}}|c| = \left|\frac{1}{2^{n_o}}c\right|, \quad x \in U_o,$$

whence, in view of (1),

$$\|g(x)\| \le \left\|\frac{1}{2^{n_o}}c\right\| = \frac{1}{2^{n_o}}\|c\| < \frac{\epsilon}{1+\|c\|}\|c\| < \epsilon, \quad x \in U_o$$

This proves the continuity of g at zero which, obviously, is equivalent to the continuity of f at x_o, completing the prof.

Remark 1. In the case of scalar convex functions defined on the entire space ($D = X$, $Y = R$, $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$, $x, y \in X$, $\lambda \in [0,1]$) the above result is implicitly contained in the book of V. Barbu and Th. Precupanu [1] (Chapter II). Here the vector-valued Jensen-convex functions are the objective on which we focus.

Let $(Y, \|\cdot\|, \preceq)$ be a Banach lattice and let $P := \{u \in Y : 0 \preceq u\}$ be a positive cone in Y. Then P is closed in Y (see Birkhoff [2]) . Jointly with the Ih-Ching Hsu's Lemma given in [4] this leads to the following

Proposition. Let P be a positive cone in a Banach lattice Y. Then $u \in P$ if and only if $p(u) \preceq 0$ for all positive continuous linear functionals on Y .

(Recall that a functional is called positive provided its restriction to a positive cone is nonnegative).

Now we are in a position to prove

Theorem 2. Let X be a real linear topological Baire space and let $(Y, \|\cdot\|, \preceq)$ be a Banach lattice with a strong unit. Assume that a set $D \subset X$ is nonempty open and convex. Then any Jensen-convex function $f : D \to Y$ such that the composition $p \circ f$ is continuous for every bounded positive linear functional p on Y is continuous. In particular, every weakly continuous Jensen-convex function from D into Y is continuous.

Proof. Fix a point $(x_o, y_o) \in (D \times Y)\backslash\text{epi } f$. Then the point $z_o := y_o - f(x_o)$ does not belong to the positive cone P in Y. Therefore, according to the Proposition, a positive functional $p \in Y^*$ has to exist such that $p(z_o) < 0$. Put $\epsilon := -p(z_o)$. In view of the continuity of $p \circ f$ and p there exist neighbourhoods $U_{x_o} \subset D$ and V_{y_o} of points x_o and y_o , respectively, such that

$$|(p \circ f)(x) - (p \circ f)(x_o)| < \frac{1}{2}\epsilon \quad \text{for} \quad x \in U \tag{9}$$

and

$$|p(y) - p(y_o)| < \frac{1}{2}\epsilon \quad \text{for } y \in V_{y_o}. \tag{10}$$

The cartesian product $W_{(x_o,y_o)} := U_{x_o} \times V_{y_o}$ is contained in $D \times Y$ and yields a neighbourhood of the point (x_o, y_o). Fix arbitrarily a point $(x, y) \in W_{(x_o,y_o)}$. Then relations (9) and (10) imply the inequalities

$$p(f(x_o)) - p(f(x)) < \frac{1}{2}\epsilon \text{ and } p(y) - p(y_o) < \frac{1}{2}\epsilon$$

whence

$$p(y) - p(f(x)) + p(f(x_o)) - p(y_o) < \epsilon$$

or, equivalently,

$$p(y - f(x)) < p(y_o - f(x_o)) + \epsilon = p(z_o) + \epsilon = 0.$$

A repeated appeal to the Proposition proves that $y - f(x) \notin P$. Thus $W_{(x_o,y_o)} \subset (D \times Y)\backslash\text{epi } f$ which shows that the epigraph of f is closed in $D \times Y$. In virtue of Theorem 1 this finishes the proof.

The theorem just proved allows one to reduce the investigations of regularity properties of convex transformations to the scalar case (convex functionals). This is visualized by the following series of corollaries.

Theorem 3. Let X, Y and D have the same meaning as in Theorem 1. If $f : D \to Y$ is Jensen-convex and order-bounded above (resp. metrically bounded) on a second category Baire subset of D, then f is continuous.

Proof. Let $f(x) \preceq c$ (resp. $\|f(x)\| \leq M$) for $x \in T \subset D$, where T is of the second Baire category. Take any positive functional $p \in Y^*$. Then $\varphi := p \circ f$ is a scalar Jensen-convex function on D and $\varphi(x) \leq p(x)$, $x \in T$, because of the positivity of p (resp. $|\varphi(x)| = |p(f(x))| \leq \|p\|\|f(x)\| \leq M\|p\|$, $x \in T$). Therefore φ is continuous (see e.g. Kuczma [6]) and hence so is f by virtue of Theorem 2.

Theorem 4. If Y satisfies the assumptions of Theorem 1 and D is a nonempty open and convex subset of R^n, then any weakly measurable Jensen-convex function from D into Y is continuous.

Proof. Let $f : D \to Y$ be a weakly measurable Jensen-convex function and let $p \in Y^*$ be positive. Then $\varphi := p \circ g$ is a measurable Jensen-convex function with values in R. Thus φ is continuous (see Kuczma [6], for instance). An appeal to Theorem 2 finishes the proof.

Theorem 5. Let $(Y, \|\cdot\| \preceq)$ be a boundedly complete Banach lattice with a strong unit and let an open interval $(a, b) \subset R$ be given. Then a function $f : (a, b) \to Y$ is Jensen-convex and continuous if and only if there exists a function $g : (a, b) \to Y$ such that

$$f(s) + tg(s) \preceq f(s+t) \quad \text{for all } s \in (a, b) \quad \text{and } t \in (a-s, b-s) \tag{11}$$

Proof. Necessity. Since f is Jensen-convex and continuous we have

$$f(\lambda x + (1-\lambda)y) \preceq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } \lambda \in [0,1] \text{ and } x, y \in (a, b). \tag{12}$$

This results from (4), the density of dyadic numbers in $(0, 1)$ and the fact that the positive cone in Y is closed. Essentially the same proof like that presented in Kuczma's book [6], shows that relationship (12) forces the function

$$I(s,t) := \frac{f(s+t)-f(s)}{t}, \ s \in (a,b), \ t \in (a-s, b-s),$$

to be nondecreasing with respect to the second variable (actually, I is nondecreasing in either variable). Fix an $s \in (a,b)$ and take $u < 0 < t$ such that $s+u$ and $s+t$ are in (a,b). Then

$$I(s,u) \preceq I(s,t)$$

which shows that the set $\{I(s,t): \ t > 0, \ s+t \in (a,b)\}$ is bounded below and since the lattice Y is boundedly complete, the function

$$g(s) := \inf\{I(s,t): \ t > 0, s+t \in (a,b)\}$$

is well defined on (a,b). Plainly, $g(s) \preceq I(s,t)$ for all $s \in (a,b)$ and all $t > 0$ such that $s+t \in (a,b)$, which gives (11) for positive t.

Likewise, taking an arbitrary $s \in (a,b)$ and $t < 0 < u$ such that $s+t, \ s+u \in (a,b)$, in view of the monotonicity of I, one obtains

$$I(s,t) \preceq I(s,u).$$

Therefore, function

$$h(s) := \sup_{t<0}\{I(s,t): \ t < 0, \ s+t \in (a,b)\}, \ s \in (a,b), \tag{13}$$

is well defined on (a,b) and, obviously,

$$h(s) \preceq I(s,u) \text{ for all } s \in (a,b) \text{ and all positive } u \in (a-s, b-s).$$

Consequently,

$$h(s) \preceq \inf_{u>0}\{I(s,u): \ u > 0, \ s+u \in (a,b)\} = g(s), \ s \in (a,b),$$

i.e.

(14) $$h(s) \preceq g(s) \text{ for } s \in (a,b).$$

Now, the inequality

$$I(s,t) \preceq h(s), \ s \in (a,b), \ t < 0, \ t \in (a-s, b-s),$$

resulting from (13), says that

$$f(s) + th(s) \preceq f(s+t), \ s \in (a,b), \ t < 0, \ t \in (a-s, b-s),$$

which jointly with (14) implies that

$$f(s) + tg(s) \preceq f(s+t), \ s \in (a,b), \ t < 0, \ t \in (a-s, b-s).$$

Thus (11) is satisfied for all $s \in (a,b)$ and $t \in R$ such that $s+t$ is in (a,b) (for $t = 0$ inequality (11) is trivially satisfied).

Sufficiency. Suppose that f satisfies (11) with some function $g(: (a,b) \to Y$ and fix a positive functional $p \in Y*$. Put $\varphi = p \circ f$ and $\psi := p \circ g$. Then

(15) $$\varphi(s) + t\psi(s) \leq \varphi(s+t), \ s \in (a,b), \ t \in (a-s, b-s).$$

Then φ is Jensen-convex and ψ is increasing (see Hardy-Littlewood-Polya [3] on Ih-Ching Hsu [4]). Fix a point $c \in (a,b)$ and put $t := c - s$ in (15). Then

$$\varphi(s) \leq \varphi(c) + (s-c)\psi(s), \ s \in (a,b),$$

i.e. φ is bounded above by a measurable function on (a,b) which proves that φ is continuous (see Kuczma [6]), for example). Thus $p \circ f$ is continuous which, by virtue of Theorem 2, in view of the unrestricted choice of p, gives the continuity of f and finishes the proof.

Remark 2. To prove the sufficiency we did not use the assumption that Y is boundedly complete. Moreover, one can show even more about f in this direction giving the integral representation of f (Ih-Ching Hsu [4]) but the proof is long and involved. Our goal here was to give yet another application of Theorem 2. On the other hand, Theorem 5, as phrased in the form of a necessary and sufficient conditin, gives an affirmative answer to a question raised by Ih-Ching Hsu in [4] whether vector-valued convex functions can be characterized by (11).

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