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CONVEXITY WITH GIVEN INFINITE WEIGHT SEQUENCES

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ABSTRACT

The aim of the present note is to investigate the funcional inequality

$$f(\sum_{i=1}^{\infty}\lambda_i x_i) \le \sum_{i=1}^{\infty}\lambda_i f(x_i),$$

where $f : D \to R_+$ is an unknown function, $D \subset R^n$ is a compact convex set and (λ_n) is a fixed decreasing sequence with $\lambda_n > 0$ and $\lambda_1 + \lambda_2 + \cdots = 1$, further this inequality holds for all sequences (x_n) from D.

1. In his paper [1] C. Alsina investigated the inequality

$$T(\sum_{i=1}^{\infty}\frac{a_i}{2^i}, \sum_{i=1}^{\infty}\frac{b_i}{2^i}) \le \sum_{i=1}^{\infty}\frac{1}{2^i}T(a_i, b_i) \tag{1}$$

where $T : [0,1]^2 \to [0,1]$ and (1) holds for any sequences (a_i) and (b_i) from $[0,1]$. His result is the following

Theorem A. If T satisfies (1) and $T(0,0) = T(0,1) = T(1,0) = 0$ then

$$T(x,y) \le \max(x+y-1, 0)T(1,1) \quad x, y \in [0,1]. \tag{2}$$

In this paper we consider a generalization of the inequality (1). As a consequence of our results it turns out that (1) is equivalent to the convexity of T, and thus we obtain a new proof for the theorem of Alsina.

2. Let $D \subseteq R^n$ be a compact convex set. Denote by Λ the set of all those real sequences $\lambda := (\lambda_n)$, for which $\lambda_n \geq \lambda_{n+1} > 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ are satisfied. If $x_i \in D$ $(i \in N)$ then $\sum_{i=1}^{N} \lambda_i x_i + \left(\sum_{i=N+1}^{\infty} \lambda_i \right) x_0 \in D$ for some $x_0 \in D$, hence by the compactness of D

$$\sum_{i=1}^{\infty} \lambda_i x_i \in D.$$

Let $A(D, \lambda)$ denote the set of all those functions $f : D \to R_+$ $(R_+ := [0, \infty))$ which satisfy the inequality

$$f(\sum_{i=1}^{\infty} \lambda_i x_i) \leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \tag{3}$$

for any sequence (x_i) from D, where $\lambda \in \Lambda$ is a fixed sequence. The right hand side of (3) is either convergent, or else, in the extended set of real numbers, it is $+\infty$.

Lemma 1. Let $f : D \to R$ be an α -convex function for some fixed $\alpha \in]0, 1[$, i.e. assume that

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \tag{4}$$

holds for all $x, y \in D$. Then f is also Jensen-convex, i.e. (4) is satisfied with $\alpha = \frac{1}{2}$.

Proof. By a repeated use of the inequality (4) we get

$$\begin{aligned} f(\frac{x+y}{2}) &= f(\alpha(\alpha \frac{x+y}{2} + (1-\alpha)x) + (1-\alpha)(\alpha y + (1-\alpha)\frac{x+y}{2})) \leq \\ &\leq \alpha f(\alpha \frac{x+y}{2} + (1-\alpha)x) + (1-\alpha) f(\alpha y + (1-\alpha)\frac{x+y}{2}) \leq \\ &\leq (\alpha^2 + (1-\alpha)^2) f(\frac{x+y}{2}) + 2\alpha(1-\alpha)\frac{f(x)+f(y)}{2} \end{aligned}$$

whence

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2} \tag{5}$$

for any $x, y \in D$.

Remarks. (1) The statement of the above lemma is an immediate consequence of a well-known result of Kuhn [6] stating that if (4) is valid for some fixed $\alpha = \alpha_0 \in]0,1[$ then it is valid also for any element of the intersection $K \cap [0,1]$, where K is the smallest subfield of R containing α_0. However, our proof of Lemma 1 -differing in this form the proof of Kuhn- does not use transfinite means, it is quite elementary.

(2) It is known that the Jensen-convexity of f, i.e., the validity of (5) implies the validity of (4) for any rational number $\alpha \in [0,1]$ and $x, y \in D$. (This statement also follows from the result of Kuhn mentioned above).

Corollary. If $\lambda \in \Lambda$ and $f \in A(D,\lambda)$ then f is Jensen-convex on D.

Proof. Let $x_1 := x$ and $x_2 := x_3 := \cdots = y$ in (3). Then we see that (4) is satisfied with $\alpha = \lambda_1$. Thus, by Lemma 1, f is Jensen-convex.

For the purpose of characterizing the elements of the class of functions $A(D,\lambda)$ we shall need the following notion and result.

Definition. A sequence $\lambda \in \Lambda$ is said to be interval-filling if for any $t \in [0,1]$ there exists a sequence $\epsilon = (\epsilon_1, \epsilon_2, ...)$ from $\{0,1\}^N$, such that

$$t = \langle \epsilon, \lambda \rangle := \sum_{i=1}^{\infty} \epsilon_i \lambda_i.$$

The sequence $\lambda_n := \frac{1}{2^n}$ is clearly interval-filling.

Theorem B. ([3], [4], [7]). A sequence $\lambda = (\lambda_n) \in \Lambda$ is interval-filling if and only if

$$\lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i$$

holds for any $n \in N$.

Lemma 2. If $\lambda \in \wedge$ then any number $t \in [0,1]$ can be written in the form

$$t = \sum_{i=1}^{\infty} \alpha_i \lambda_i,$$

where $\alpha_i \in Q \cap [0,1]$ for all $i \in N$.

Proof. If $\lambda \in \wedge$ is an interval-filling sequence then we choose $\alpha_i \in \{0,1\}$ and the proof is complete. If λ is not interval-filling, then there exists a sequence $n_1 \leq n_2 \leq n_3 \leq \cdots$ of natural numbers, such that

$$\frac{\lambda_k}{n_k} \leq \sum_{i=n+1}^{\infty} \lambda_i$$

for any $k \in N$. Hence for the sequence

$$\mu := \Big(\underbrace{\frac{\lambda_1}{n_1}, \dots, \frac{\lambda_1}{n_1}}_{n_1 - \text{times}}, \underbrace{\frac{\lambda_2}{n_2}, \dots \frac{\lambda_2}{n_2}}_{n_2 - \text{times}}, \dots\Big)$$

we have $\mu \in \wedge$ and this μ is interval-filling. Thus any $t \in [0,1]$ can be written in the form

$$t = \sum_{i=1}^{\infty} \epsilon_i \mu_i$$

where $\epsilon_i \in \{0,1\}\,(i \in N)$. From this

$$t = \sum_{i=1}^{\infty} \frac{m_i}{n_i} \lambda_i$$

follows with $m_i \in \{0,1,...,n_i\}\,(i \in N)$. If we still put $\alpha_i := \dfrac{m_i}{n_i}$ then we get the statement of the lemma.

Now we are able to state the main result of this paper.

Theorem 1. Let $\lambda \in \wedge$. Then $f \in A(D,\lambda)$ holds if and only if f is convex on the set D.

Proof. Let $f \in A(D, \lambda)$. By the Corollary of Lemma 1, f is Jensen-convex, hence (4) holds for any $\alpha \in Q \cap [0,1]$ and $x, y \in D$. By Lemma 2 there exist rational numbers $\alpha_i \in [0,1]$, such that

$$t = \sum_{i=1}^{\infty} \alpha_i \lambda_i.$$

Now

$$1 - t = \sum_{i=1}^{\infty} (1 - \alpha_i)\lambda_i.$$

Hence

$$f(tx + (1-t)y) = f(\sum_{i=1}^{\infty} \alpha_i \lambda_i x + \sum_{i=1}^{\infty} (1-\alpha_i)\lambda_i y) =$$

$$= f\left(\sum_{i=1}^{\infty} \lambda_i(\alpha_i x + (1-\alpha_i) y) \right) \leq \sum_{i=1}^{\infty} \lambda_i f(\alpha_i x + (1-\alpha_i)y) \leq$$

$$\leq \sum_{i=1}^{\infty} \lambda_i \alpha_i f(x) + \sum_{i=1}^{\infty} \lambda_i (1-\alpha_i) f(y) = tf(x) + (1-t)f(y),$$

i.e. f is convex.

To prove the reverse statement, assume that f is a convex function on D and let $(x_n) \subset D$ be an arbitrary sequence. Denote by X the affine hull of the sequence (x_n). Assume that X is a k-dimensional hyperplane $(0 \leq k \leq n)$. Thus, without loss of generality, we may assume that it is spanned by $x_1, \cdots, x_{k+1}$. Then $x_0 := \left(\frac{\lambda_1}{\mu}\right) x_1 + \cdots + \left(\frac{\lambda_{k+1}}{\mu}\right) x_{k+1}$ (where $\mu := \lambda_1 + \cdots + \lambda_{k+1}$) is an interior point of the convex hull of $x_1, \cdots, x_{k+1}$, i.e., x_0 is an interior point of $D \cap X$ in the k-dimensional topology. On the other hand, $D \cap X$ is a compact convex set, thus $x* := \sum_{i=k+2}^{\infty} (\frac{\lambda_i}{1-\mu}) x_i \in D \cap X$. Therefore $\mu x_0 + (1-\mu)x* = \sum_{i=1}^{\infty} \lambda_i x_i$ is also an interior point of $D \cap X$ in the k-dimensional topology. Then the convexity of f implies that $f|_{D \cap X}$ is continuous at the point $\sum_{i=1}^{\infty} \lambda_i x_i$ (since $f|_{D \cap X}$ is a convex function and convex functions are continuous at interior points of their domain.)

Now we can prove (3). By the convexity of $f|_{D \cap X}$ we have

$$f\left(\sum_{i=1}^{N} \lambda_i x_i + \left(\sum_{i=N+1}^{\infty} \lambda_i \right) x_0 \right) \leq \sum_{i=1}^{N} \lambda_i f(x_i) + \sum_{i=N+1}^{\infty} \lambda_i f(x_0)$$

for all $\mathcal{N} \in N$. Taking the limit $\mathcal{N} \to \infty$ and using the continuity of $f|_{D\cap X}$ at $\sum_{i=1}^{\infty} \lambda_i x_i$ we obtain (3).

The proof is complete.

Applying Theorem 1 we can give a new proof for Theorem A. Assume that T satisfies (1). Then, by Theorem 1, T is a convex function. Thus Theorem A follows from the following

Theorem 2. If $T := [0,1]^2 \to [0,1]$ is convex and $T(0,0) = T(0,1) = T(1,0)$, then (2) is satisfied.

Proof. Let $e_0 := (0,0)$, $e_1 := (1,0)$, $e_2 := (0,1)$. Then $T(e_0) = T(e_1) = T(e_2) = 0$. If $x \in [0,1]^2$ then there are two possibilities:

(i) x is an element of the closed triangular region determined by e_0, e_1, e_2.

(ii) x is in the complement with respect to $[0,1]^2$ of the triangular region defined in (i).

In case (i) we can write $x = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2$, where $\lambda_i \geq 0$ and $\lambda_0 + \lambda_1 + \lambda_2 = 1$. Hence, by the convexity of T,

$$T(x) \leq \lambda_0 T(e_0) + \lambda_1 T(e_1) + \lambda_2 T(e_2) = 0. \tag{6}$$

In case (ii) we have $x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e$ with $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Hence by the convexity of T

$$T(x) \leq \lambda_1 T(e_1) + \lambda_2 T(e_2) + \lambda_3 T(e) = \lambda_3 T(e). \tag{7}$$

In case (i) $x = (x_1, x_2)$ satisfies $x_1 + x_2 - 1 \leq 0$; in case (ii) we get $x_1 + x_2 - 1 > 0$. Hence in case (ii) $x = (x_1, x_2) = (\lambda_1 + \lambda_3, \lambda_2 + \lambda_3)$, whence

$$x_1 + x_2 - 1 = \lambda_1 + \lambda_2 + 2\lambda_3 - 1 = \lambda_3.$$

Therefore (7) implies

$$T(x) \leq (x_1 + x_2 - 1)T(e). \tag{8}$$

The inequalities (6) and (8) together yield (2).

Remark. By putting $a_1 = x_1$, $a_2 = a_3 = \cdots = x_2$, $b_1 = y_1$, $b_2 = b_3 = \cdots = y_2 in (1)$, we can see that T is Jensen convex. On the other hand, by our hypotheses on T, we have that $0 \leq T(x) \leq 1$, i.e. T is a bounded function. Then, by the theorem of Bernstein-Doetsch ([2], [5]), T is convex. From this we see that the convexity of T follows already form a special case of (1) and the boundedness of T :

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