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ALGEBRAIC METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS

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SUMMARY

By means of the reduction of boundary value problems to algebraic ones, conditions for the existence of solutions and explicit expressions of them are obtained. These boundary value problems are related to the second order operator differential equation $X^{(2)}+A_1X^{(1)}+A_oX=0$, *and* $X^{(1)}=A+BX+XC$. *For the finite-dimensional case, computable expressions of the solutions are given.*

1. Introduction.

For the finite-dimensional case, second order operator differential equations are important in the theory of damped oscillatory systems and vibrational systems, [4], [7]. Infinite-dimensional equations occur frequently in the theory of stochastic processes, the degradation of polymers, infinite ladder network theory in engineering,[1],[18], denumerable Markov chains, and moment problems, [14],[21]. Throughtout this paper H will denote a complex separable Hilbert space and L(H) will denote the algebra of all bounded linear operators on H. In the section 2 of this paper we consider two-point boundary value problems of the type

$$\ddot{U} + A_1\dot{U} + A_oU = 0$$
$$F_1\dot{U}(a)-U(0)G_1 = E_1 \qquad (1.1)$$
$$F_2\dot{U}(b)-U(0)G_2 = E_2$$

where the coefficient are operators in $L(H)$, $0<a\leqslant b$, $0\leqslant t\leqslant b$.

It is well known that the resolution problem of a scalar second linear differential equation with constant coefficients, $\ddot{u} + a_1\dot{u} + a_ou=0$, is solved from the determination of the roots of the associated algebraic equation, $z^2+a_1z+a_o=0$. In an analogous way to the scalar case we obtain sufficient conditions for the resolution problem (1.1) in terms of solutions of the algebraic operator equation

$$T^2 + A_1T + A_o = 0 \qquad (1.2)$$

This equation has been studied in [13] where a methodology for its resolution is given. The problem of the solvability of the equation (1.2) is related to the factorization of the polynomial operator $L(z)=z^2+zA_1+A_o$. For the finite-dimensional case, it is known, [7], that if the companion operator

$$C_L = \begin{bmatrix} 0 & I \\ -A_o & -A_1 \end{bmatrix}$$

is diagonable, then there is a linear factorization $L(z)=(zI+T_1)(zI+T_2)$, and in this case $-T_2$, is a solution of (1.2). Different conditions for the existence of a linear factorization of $L(z)$ are given in [9] for the finite-dimensional case, and in [20] for the infinite-dimensional one.

In a recent paper, [12], we gave a resolution method for solving boundary value problems of the type

$$\begin{aligned} \dot{U} &= A + BU + UC \\ E_1U(b)-U(0)F_1 &= G_1 \qquad (1.3) \\ E_2U(a)-U(0)F_2 &= G_2 \end{aligned}$$

in terms of the solutions of an algebraic Lyapunov system of the type

$$\begin{aligned} A_1+B_1X + XC_1 &= 0 \\ A_2+B_2X + XC_2 &= 0 \end{aligned} \qquad (1.4)$$

where the coefficient matrices arising in (1.4) are related to the data problem. Moreover, in [12], a resolution method for solving the system (1.4) by application of the annihilating polynomial technique is given. In the section 3 we obtain an existence and uniqueness condition for the system (1.4) and an explicit expression of the solution of (1.3) under the uniqueness condition is given.

For the sake of clarity in the exposition we recall some concepts that will be used below. If T lies in L(H), its spectrum $\sigma(T)$ is the set of all complex numbers z such zI-T, is not invertible in L(H), $\sigma_\pi(T)$ denotes its approximate point spectrum and $\sigma_\delta(T)$ denotes its approximate defect spectrum. Definitions and properties of these parts of the spectrum $\sigma(T)$ can be found in [11].

For a matrix $A \in C_{nxn}$, where C denotes the complex plane, we represent by A^- as a matrix satisfying $A A^- A = A$. We recall that the Drazin inverse of A, denoted by A^D satisfies this property and its computation is available as a polynomial in A, see [5], [19], for details.

If C is a mxm real matrix, $C\varepsilon R_{mxm}$ and $D\varepsilon R_{kxs}$, then the tensor product of C and D written $C \otimes D$, is defined as the partitioned matrix

$$C\otimes D=\begin{bmatrix} c_{11}D, & \dots, & c_{1n}D \\ \vdots & & \vdots \\ c_{m1}D, & \dots, & c_{mn}D \end{bmatrix}$$

an account of the uses and applications of the operation $\otimes$ can be found in [15] or [16]. If $C\varepsilon R_{nxn}$, we denote

$$\hat{C}=\mathrm{vec}C=\begin{bmatrix} C_{.1} \\ \vdots \\ C_{.n} \end{bmatrix} \quad , \quad C_{.j}=\begin{bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{bmatrix}.$$

If M, N and P are matrices with suitable dimensions and P^T denotes the transposed matrix of P, then for the column lemma, [3], one gets $\mathrm{vec}(MNP)=(P^T\otimes M)\mathrm{vec}N$.

2. Boundary value problems for the second order operator differential equation.

For the sake of clarity in the exposition we state the following result whose proof can be found in [13].

<u>Lemma 1. ([13]).</u> If T_o is a solution of the operator equation (1.2), then the operator function

$$U(t)=\exp(tT_o)C_o+\int_o^t \exp((t-s)T_o)\exp(sT_1)Dds \quad (2.1)$$

where $T_1=-(T_o+A_1)$ and $D=C_1-T_oC_o$, is a solution of the problem

$$\ddot{U} + A_1\dot{U} + A_oU = 0$$
$$U(0)=C_o; \quad \dot{U}(0) = C_1 \qquad (2.2)$$

Our first result gives an algebraic condition on the data problem for solving the boundary value problem (1.1).

Theorem 1. Let us consider the boundary value problem (1.1). If T_o is a solution of the equation (1.2) and the system

$$AT - TB = C$$
$$DT - TE = F \qquad (2.3)$$

is compatible, where

$$T_1= -(T_o+A_1); \quad A=F_1\exp(aT_o); \quad B=G_1; \quad C=E_1-F_1\exp(aT_1)$$
$$D=F_2T_o\exp(bT_o); \quad E=G_2; \quad F=E_2-F_2\exp(bT_1) \qquad (2.4)$$

then the problem (1.1) is solvable and a solution is given by the expression

$$U(t)=\exp(tT_o)C_o; \quad 0 \leq t \leq b \qquad (2.5)$$

where C_o is a solution of the system (2.3)-(2.4).

Proof. Given the operator T_o, from the lemma 1, it follows that a solution of the problem (2.1) has the expression (2.2). Taking $C_1=T_oC_o$, one gets D=0 and

$$U(t)=\exp(tT_o)C_o; \quad t\varepsilon[0,b] \qquad (2.6)$$

where $C_o=U(0)$. By differentiation in (2.6), it follows that

$$\dot{U}(t)=\exp(tT_o)T_oC_o=T_o\exp(tT_o)C_o=T_oU(t) \qquad (2.7)$$

From here, the boundary value conditions arising in (1.1) are verified if there exists a solution C_o of the system

$$(F_1T_o\exp(aT_o))C_o - C_oG_1 = E_1$$
$$(F_2T_o\exp(bT_o))C_o - C_oG_2 = E_2 \tag{2.8}$$

From the hypothesis the result is established.

Remark 1. A methodology for solving infinite-dimensional systems of the type (2.3) by means of its reduction to another one of the type TE=F, MT=N, by application of the technique of annihilating analytic functions of operators is suggested in [10].

The following result is concerned with the boundary value problem (1.1), when a=b, $F_1=F_2=F$, $G_1=G_2=G$ and $E_1=E_2=E$.

Corollary 1. Let us consider the problem

$$\ddot{U} + A_1\dot{U} + A_oU = 0$$
$$F\,\dot{U}(b)-U(0)G=E \tag{2.9}$$
$$0< t < b$$

If T_o is a solution of the equation (1.2) and

$$\sigma_\delta(FT_o\exp(aT_o)) \cap \sigma_\pi(G)=\emptyset \tag{2.10}$$

then the problem (2.9) is solvable and a solution is given by the expression

$$U(t)=\exp(tT_o)C_o;\quad t\in[0,b] \tag{2.11}$$

where C_o is a solution of the algebraic equation

$$AT - TB = C \tag{2.12}$$

and

$$A=FT_o\exp(aT_o);\ B=G;\ C=E-F\exp(aT_1);\ T_1=-(T_o+A_1) \quad (2.13)$$

Proof. From the hypothesis (2.10) and th. 5 of [6], there exists a solution C_o of the equation (2.12). From here and from theorem 1 the result is proved.

In order to obtain an explicit solution of (2.11) it is clear we need an explicit expression of the solutions of the algebraic problem (2.12). In [10], conditions for obtaining such expressions are given. In [1] and [8] the solutions of this equation when the coefficient operators belong to certain classes are studied. For the finite-dimensional case an explicit and computable expression of the solutions of (2.12) is available.

Corollary 2. If H is a finite-dimensional Hilbert space and substituting the hypothesis (2.10) by the following one

$$\sigma(FT_o\exp(aT_o)) \cap \sigma(G)=\emptyset$$

then a solution of (2.9) is given by (2.11) being C_o the operator

$$\left(\sum_{k=o}^{n} p_k (FT_o\exp(aT_o)^k)^{-1}\left(\sum_{k=1}^{n}\sum_{j=1}^{k} p_j (FT_o\exp(aT_o))^{j-1}EG^{k-j}\right)\right.$$

and $p(z)=\sum_{k=o}^{n} p_k z^k$, the characteristic polynomial of G.

Proof. The result is a consequence of the corollary 1 and [10].

Next result is concerned with a particular case of the problem (1.1) that is interesting in the applications and for which an explicit and computable expression of its solution is available.

Theorem 2. Let H be a finite-dimensional Hilbert space and let T_o be a solution of the algebraic equation (1.2). If we consider the boundary value problem

$$\ddot{U} + A_1\dot{U} + A_oU = 0$$

$$U(0)G_2=E_2;F_1\dot{U}(a)=E_1 \qquad (2.14)$$

$$0 < t < b$$

then this problem is solvable if the following condition is satisfied

$$F_1\exp(aT_o)T_oE_2=E_1G_2 \qquad (2.15)$$

In this case the matrix function $U(t)=\exp(tT_o)C_o$, satisfies (2.14) for every matrix C_o of the type

$$C_o=A^-E_1+E_2G_2^- - A^-AE_2G_2^-+(I-A^-A)V(I-G_2G_2^-) \qquad (2.16)$$

where V is an arbitrary matrix with dim(H)=dim V, and $A=F_1\exp(aT_o)T_o$.

Proof. Considering the theorem 1 with $G_1=0=F_2$, it follows that (2.14) is solvable if there exists a solution C_o of the matrix system

$$A\,C_o = E_1$$
$$C_oG_2 = E_2 \qquad (2.17)$$

where $A=F_1\exp(aT_o)T_o$, and in this case the matrix function $U(t)=\exp(tT_o)C_o$, is a solution of (2.14). Now, from the lemma 2.2 of [17] and the hypothesis (2.15), the system (2.17) is solvable and its general solution is given by (2.16). From here the result is established.

3. Boundary value problems for the Lyapunov equation $\dot{U} = A+BU+UC$.

In this section we are interested with the determination of explicit and computable expressions for solutions of finite-dimensional boundary value problems related to the Lyapunov type equation $\dot{U} = A+B\ U+UC$.

<u>Lemma 2.</u> Let A_i, B_i and C_i be matrices in R_{nxn}, for $i=1,2$, and let us denote by $r_{i,j}$, for $1 \leqslant j \leqslant m_i$, $i=1,2$ and $s_{i,k}$, for $1 \leqslant k \leqslant n_i$, $i=1,2$, respectively the eigenvalue sets of the matrix B_i and C_i respectively. If the following property is satisfied

$$r_{i,j}+s_{i,k}\neq 0,\quad 1 \leqslant j \leqslant m_i,\quad 1 \leqslant k \leqslant n_i,\quad i=1,2 \qquad (3.1)$$

then a necessary and sufficient condition for the existence and uniqueness of solutions of the system (1.4) is the following one

$$(I \otimes B_1+C_1^T \otimes I)^{-1}\hat{A}_1 = (I \otimes B_2+C_2^T \otimes I)^{-1}\hat{A}_2 \qquad (3.2)$$

In this case the only solution of (3.2) is given by

$$\hat{X} = (I \otimes B_i+C_i^T \otimes I)^{-1}\hat{A}_i,\quad i=1,2 \qquad (3.3)$$

<u>Proof.</u> By application of the tensor product to each equation of the system (1.4), and taking into account the column lemma, [3], one gets

$$\hat{A}_i + (I \otimes B_i+C_i^T \otimes I)\hat{X} = \hat{0},\quad i=1,2 \qquad (3.4)$$

From the hypothesis (3.1) it follows that the matrices $I \otimes B_i+C_i^T \otimes I$, are invertible for $i=1,2$. From here and (3.4) the result is obtained.

Theorem 3. Let us consider the problem (1.1) and let A_i, B_i and C_i be the matrices defined by the expressions

$$B_1=F_1\exp(aB),\quad C_1=-G_1\exp(-Ca),\quad B_2=F_2\exp(Bb),\quad C_2=-G_2\exp(-Cb)$$

$$A_1=-E_1\exp(-Ca)+F_1\int_o^a \exp(B(a-s))A\exp(C(t-s))ds \tag{3.5}$$

$$A_2=-E_2\exp(-Cb)+F_2\int_o^b \exp(B(b-s))A\exp(-Cs)ds$$

If the spectral condition (3.1) is satisfied then the boundary value problem (1.3) has only one solution given by the expression

$$U(t)=\exp(Bt)C_o\exp(Ct) + \int_o^t \exp(B(t-s))A\exp(C(t-s))ds \tag{3.6}$$

where C_o is given by the expression (3.3).

Proof. From theorem 6 of [12], the boundary value problem (1.3) is solvable if an only if the system (1.4)-(3.5) is solvable. Furthermore, the solution set for the problem (1.3) is given by (3.6), where C_o is a solution of the algebraic system (1.4)-(3.5). Now, the result is a consequence of the previous lemma 2 and theorem 6 of [12].

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