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WEAKLY PROJECTABLE AND PROJECTABLE W-ALGEBRAS

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ABSTRACT

In this paper we introduce and study the weakly projectable and projectable W-algebras and we review the Representatioon Theorem of [1].

Definition 1. A W-algebra A is weakly projectable if and only if for all a,b in A $a^{\nabla} v_F b^{\nabla} = a^{\nabla} v_P b^{\nabla}$.

We can give also

Definition 1'. A W-algebra A is weakly projectable if and only if for all a,b in A $a^{\nabla} v_F b^{\nabla} = (avb)^{\nabla}$.

Let WP be the class of all weakly projectable W-algebras; let us characterize now its elements

Proposition 1. Let A be a W-algebra. $(A \in WP)$ iff $(\forall p \in \mathrm{Spec}A\ \exists! q \in \pi A : q \subseteq p)$.

Proof of the direct implication: Let $p \in \mathrm{Spec}A$ and $N = \cap\{q \in \pi A : q \subseteq p\}$. We've showed in [1], prop. (4.8), that $N = \cup\{f^{\nabla} : f \in A \setminus p\}$; hence it is sufficient to prove that N is a prime implicative filter. So, let $a,b \in A \setminus N$; as $a \notin N$ then there exists $q \in \pi A$ $q \subseteq p$ such that

$a \notin q$ and since $q \supseteq (u) = a^{\nabla} \cap a^{\nabla\nabla}$ then we have $q \supseteq a^{\nabla}$ and therefore $p \supseteq a^{\nabla}$. In the same way $p \supseteq b^{\nabla}$ and thus $p \supseteq a^{\nabla} \vee_F b^{\nabla} = (a \vee b)^{\nabla}$. Consequently $a \vee b \neq u$ as we wanted to prove.

Proof of the inverse implication: Let $a,b \in A$. It is clear that $a^{\nabla}\vee_F b^{\nabla} \subseteq (a\vee b)^{\nabla}$; so, it is sufficient to prove that $(a\vee b)^{\nabla} \subseteq a^{\nabla}\vee_F b^{\nabla}$. Then let $f \in A\setminus(a^{\nabla}\vee_F b^{\nabla})$; we can consider p, maximal in the family of implicative filters not containing f which extend $a^{\nabla}\vee_F b^{\nabla}$. Such a p is prime. By hypothesis p extends a single element of πA, say q, which, by (4.8) of [1], coincides with $\cup\{\xi^{\nabla}:\xi \in A\setminus p\}$. We have $p \supseteq a^{\nabla}$. Observe that if $a \in q$ then there exisits $\xi \in A\setminus p$ such that $a \in \xi^{\nabla}$ and therefore $p \supseteq a^{\nabla} \supseteq \xi^{\nabla\nabla} \supseteq \{\xi\}$ which contradicts the fact that $\xi \in A\setminus p$. So, $a \notin q$. In the same way, $p \supseteq b^{\nabla}$ and therefore $b \notin q$. Hence $a \vee b \notin q$ and $f \notin (a \vee b)^{\nabla}$ as we wanted to prove.

In the following theorem we give a characterization of weakly projectable W-algebras relative to the structure of fibers on the Representation Theorem (RT) of [1].

<u>Theorem 1.</u> On the global subdirect representation of a W-algebra A

$$\alpha: A \to \otimes\,(A/F_m : m \in \mu A) \quad \text{-see [1], RT-}$$

the fibers A/F_m are chains if and only if A is weakly projectable.

An implication is trivial: if A is weakly projectable then F_m is the single prime minimal implicative filter which is contained in m and, obviously, F_m is prime and therefore the quotient algebras A/F_m are chains.

We are going to prove the other implication:

If for all $m \in \mu A$, A/F_m is a chain then for all $m \in \mu A$, $F_m = \cap\{q \in \pi A: q \subseteq m\}$ is an element of SpecA and thus, each maximal (a fortiori, each prime) implicative filter extends a single prime minimal implicative filter and this is equivalent to the weakly projectability of A.

In the remainder of this paper, A will be a weakly projectable W-algebra. It is clear (in the context of the weakly projectable W-algebras) the existence of a natural bijection between πA and μA. In the previously mentioned RT of [1], the F_m run over all the minimal spectrum of A. This fact suggest us the study of the representation of A by the minimal spectrum himself, endowed this one with a suitable topology.

We consider the application $\underline{m}$: $\mu A \to \pi A$ where $\underline{m}(m)$ is the single element of πA which is contained in m.

Definition 2. The subspectral topology on πA (A weakly projectable) is the quotient topology on πA with respect to the application $\underline{m}$ (μA endowed with the natural spectral topology).

Terminology. Let $0 \subseteq \pi A$. We say that 0 is an open set if and only if 0 is open with respect to the spectral topology on πA, and we say that 0 is a S-open set if and only if 0 is open considering on πA the subspectral topology.

Characterize now the S-opens of πA:

$0 \subseteq \pi A$ is S-open iff $\exists U$ open set of μA : $0 = \underline{m}(U)$ iff
$\exists F \in I_F(A) : 0 = \underline{m}(\mu A \cap S(F))$ iff $0 = \{p \in \pi A : \exists m \in \mu A\ m \not\supseteq F$ and $p = \underline{m}(m)\}$

If we denote with Mp the only maximal implicative filter whichs extends p ($p \in \mathrm{Spec}A$) then we have

Proposition 2. $0 \subseteq \pi A$ S-open iff $\exists F \in I_F(A) : 0 = \{p \in \pi A : Mp \not\supseteq F\}$

To shorten, we refer to the set $\{p \in \pi A : Mp \not\supseteq F\}$ as 0_F. If f is an element of A, we write 0_f instead of $0_{F<f>}$. With these notations we have that the S-opens on πA are exactly the 0_F for $F \in I_F(A)$.

It is not difficult to prove the following

Proposition 3. $\forall F \in I_F(A)$ $O_F = \cup\{O_f : f \in F\}$

Corollary. The family $\langle O_f : f \in A\rangle$ is a basis for the subspectral topology on πA.

We relate now this concept of subspectral topology with the concept of spectral topology (on πA).

Proposition 4. For all $f \in A$, O_f is an open set.

Proof. Let $p \in O_f$. We have $f \notin M_p$ and thus $Mp \in S(f)$. Mp is a maximal implicative filter and, as we've show in [1], prop. 4.2, there exists C, closed neighborhood of Mp in SpecA such that $C \subseteq S(f)$. Since the family $\{S(g) : g \in A\}$ is a basis for the spectral topology on SpecA, there exists $g \in A$ such that $Mp \in S(g) \subseteq Int(C) \subseteq C \subseteq S(f)$. As $Mp \in S(g)$ and $p \subseteq Mp$ then we have -see [1] prop (4.3)- $p \in S(g)$. It is easy to see that $S(g) \cap \pi A \subseteq O_f$ and therefore $p \in Int(O_f)$. We are then showed that each point of O_f is an interior point, that is, the O_f are open sets, as we wanted to prove.

We introduce now the concept of projectability

Definition 3. A W-algebra A is projectable if and only if the following conditions hold:

(i) A is weakly projectable

(ii) The spectral and subspectral topologies on πA are the same.

Denote with $\mathcal{P}$ the class of all projectable W-algebras. We've already proved the following

Theorem 2. If $A \in \mathcal{P}$ then the natural representation

$A \to \otimes (A/p : p \in \pi A)$ -πA endowed with the natural spectral topology- is a global subdirect representation.

The remainder of this paper is devoted to prove that the last theorem characterizes in fact the projectable W-algebras.

Proposition 5. Let A be a W-algebra (weakly projectable, of course). If the natural application $A \to \otimes (A/p : p \in \pi A)$ -πA endowed with the natural spectral topology- is a global subdirect representation, then for each $a \in A, a^{\nabla} \vee_F a^{\nabla\nabla} = A$.

The proof of the present proposition is divided in lemmas.

Lemma 1. For all $a \in A$, the equalizer $[a=u] = \{p \in \pi A : a \in p\}$ is a clopen set.

Indeed, $[a=u]$ is open and therefore there exist $F \in I_F(A)$ such that $[a=u] = \pi A \cap S(F)$. This implies that $\pi A \cap S(A) = \pi A \cap S(F)^c$ and thus $\pi A \cap S(A) = [a \neq u]$ is a clopen set on πA. Obviously, $[a=u]$ is also a clopen set.

Lemma 2. Let $a \in A$. The element $a' = (a'_p : p \in \pi A) \in \otimes (A/p : p \in \pi A)$ Defined by

$$a'_p = [u]_p \text{ if } p \in [a=u] \text{ and } a'_p = [0]_p \text{ if } p \notin [a=u]$$

is also an element of A.

It is sufficient to show that for $x \in A$ the set $[a'=x]$ is an open set, but this is obvious because

$$[a'=x] = ([a'=u] \cap [a'=x]) \cup ([a'=0] \cap [a'=x]) = ([a=u] \cap [x=u]) \cup ([a \neq u] \cap [x=0]).$$

Proof of the proposition 5. We have $a' \wedge \neg a' = 0$. It is not difficult to prove that $a' \in a^{\nabla\nabla}$ and $\neg a' \in a^{\nabla}$. Consequently, $0 \in a^{\nabla} \vee_F a^{\nabla\nabla}$ and thus $a^{\nabla} \vee_F a^{\nabla\nabla} = A$ as we wanted so show.

Proposition 6. If for all $a \in A$, $a^{\nabla} \vee_F a^{\nabla\nabla} = A$ then each prime implicative filter of A,p, extends a single prime u-filter,Z, which is $Z = \cup \{\xi^{\nabla} : \xi \in A \backslash p\}$.

Indeed, call $N = \cup\{\xi^{\nabla} : \xi \in A\backslash p\}$. Let $p \in \mathrm{Spec}A$, and let $Z \subseteq p$ be a prime u-filter. We have already proved in [1], prop. (4.8), that $N = \cap\{q \in \pi A : q \subseteq p\}$. Let's show Z=N:

If there exists $x \in Z$ such that $x \notin N$ then there exists $q \in \pi A$ $q \subseteq p$ such that $x \notin q$. Since $q \supseteq (u) = x^{\nabla} \cap x^{\nabla\nabla}$ then $q \supseteq x^{\nabla}$ and thus $p \supseteq x$. Moreover, since $x \in Z$, we have $x^{\nabla\nabla} \subseteq Z \subseteq p$ and therefore $A = x^{\nabla} \vee_F x^{\nabla\nabla} \subseteq p$. This contradiction proves $Z \subseteq N$. The other inclusion is trivial.

<u>Corollary.</u> Let A be a W-algebra. If for each $a \in A$, $a^{\nabla} \vee_F a^{\nabla\nabla} = A$ then A is weakly projectable.

<u>Proposition 7.</u> If each $p \in \mathrm{Spec}A$ extends a single prime u-filter, then the following conditions hold:

(i) For each $p \in \mathrm{Spec}A$, the mentioned primer u-filter is $N = \cup\{\xi^{\nabla} : \xi \in A\backslash p\}$

(ii) For each $a \in A$, $a^{\nabla} \vee_F a^{\nabla\nabla} = A$.

(i) is left to the reader. Proof of (ii):

If there exists $f \in A$ such that $f^{\nabla} \vee_F f^{\nabla\nabla} \neq A$, we consider $m \in \mu A : m \supseteq f^{\nabla} \vee_F f^{\nabla\nabla}$. Since $m \in \mathrm{Spec}A$, m extends a single prime u-filter $N = \cup\{\xi^{\nabla} : \xi \in A\backslash p\}$. We have $f^{\nabla\nabla} \subseteq m$ and thus (see [1], prop. (3.10)), there exists Z, prime u-filter such that $f \in Z \subseteq m$. By hypothesis, Z = N and therefore $f \in N$. This implies the existence of $\xi \in A\backslash m$ such that $\xi \in f^{\nabla} \subseteq m$ and this contradiction finish the demonstration.

<u>Proposition 8.</u> Let $f \in A$ and $p \in \pi A$. If $f^{\nabla} \vee_F f^{\nabla\nabla} = A$ then

$$f \in p \text{ is equivalent to } f^{\nabla\nabla} \subseteq Mp.$$

(trivial by using that each prime implicative filter is an u-filter).

Corollary. If $f \in A$ and $f^{\nabla} v_F f^{\nabla\nabla} = A$ then $S(f) \cap \pi A = 0_f{}^{\nabla\nabla}$

In particular, if A is a W-algebra such that for each $a \in A$ $a^{\nabla} v_F a^{\nabla\nabla} = A$, then each open set on πA is a S-open set and, by corollary of proposition-6, A is projectable.

We can summarize in the last theorem the results of the previous propositions

Theorem 3. If A is a W-algebra then the following affirmations are equivalent:

(i) A is weakly projectable + spectral and subspectral topologies on πA are the same.

(ii) The natural application $a: A \rightarrow \otimes (A/p : p \in \pi A)$ $-\pi A$ endowed with the spectral topology- is a global subdirect product representation.

(iii) For each $a \in A$, $a^{\nabla} v_F a^{\nabla\nabla} = A$.

(iv) Each prime implicative filter extends a single prime u-filter.

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