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BOUNDARY PROBLEMS FOR GENERALIZED LYAPUNOV EQUATIONS

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ABSTRACT

Boundary value problems for generalized Lyapunov equations whose coefficient are time-dependant bounded linear operators defined on a separable complex Hilbert space are studied. Necessary and sufficient conditions for the existence of solutions and explicit expressions of them are given.

1. Introduction.

Let L(H) be the linear space of all bounded linear operators on a separable complex Hilbert space H. When we endow this space with the strong topology we obtain a topological vector space that we will denote by $L_s(H)$. In this space we can look the infinite-dimensional operator Lyapunov equation

$$(d/dt)U(t)=A+BU(t)-U(t)B^*;\ U(0)=U_o \qquad (1.1)$$

where A,B,U_o and U(t) are linear operators in L(H) and B^* denotes the adjoint operator of B. Equation (1.1) arises in optimal control, [10], transport theory, [13], and filtering problems, [2]. In a recent paper, [8], it is studied the finite dimensional boundary value problem

$$(d/dt)U(t)=A+BU(t)-U(t)B^{*};\ U(b)=U(0) \tag{1.2}$$

In [11], we study the infinite-dimensional boundary value problem

$$(d/dt)U(t)=A+BU(t)-U(t)C;\ EU(b)-U(0)F=G \tag{1.3}$$

where all operators which appear in (1.3) are bounded linear operators on H. In this paper we study the infinite-dimensional boundary value problem

$$\begin{gathered}(d/dt)U(t)=A(t)+B(t)U(t)-U(t)C(t),\ 0\leqslant t\leqslant b\\ EU(b)-U(0)F=G,\end{gathered} \tag{1.4}$$

where $A(t),B(t),C(t),E,F$ and G are operators in $L(H)$. In section 2, necessary and sufficient conditions for the existence of strong solutions, that is, solutions in $L_s(H)$ of problem (1.4) are given. In addition we give explicit expressions for the solutions in terms of solutions of an algebraic operator equation associeated to the boundary value problem. We apply the results to the study of the existence of periodic solutions of the operator differential equation arising in (1.4), when the coefficient operator functions are periodic.

We recall that an operator T in L(H) whose spectrum $\sigma(T)$ is finite, is said to be an algebraic operator if there exists a polynomial $p(z)$ such that $p(T)=0$. It is clear that every finite-dimensional operator is algebraic, but for the infinite-dimensional case this fact is not true, [12], p. 53. An account of the uses and properties of algebraic operators may be found in [6].

If T is a linear operator on H with domain D(T), we denote the numerical range of T by

$$\Theta(T)= \{z\varepsilon C;\ z=(Tx,x),\ \|x\|=1\}$$

If $w\varepsilon R$, $0<\delta<\pi/2$, we denote $T_{w,\delta}=\{z\varepsilon C;\ |\arg(z-w)|\leqslant\delta+\pi/2\}$, and $\Sigma_{w,\delta}$ is the closure of the complement of $T_{w,\delta}$ in the complex plane.

Let $A(t)$ be a bounded linear operator valued function such that there exists an invertible fundamental operator $U_A(t,s)$ of the equation $(d/dt)u(t)=A(t)u(t)$, for $0\leqslant t\leqslant b$, and $0\leqslant s\leqslant t\leqslant b$. Then in accordance with the property $U_A(t,u)=U_A(t,s)U_A(s,u)$, for $0\leqslant u\leqslant s\leqslant t\leqslant b$, verified by $U_A(.,.)$, see [9] for details, and taking into account the usual notation for the finite-dimensional case, when there exists $(U_A(t,s))^{-1}$, for $b\geqslant t\geqslant s\geqslant 0$, we can extend $U_A(.,.)$ to the points $(t,s)\varepsilon[0,b]^2$, such that $0\leqslant t\leqslant s\leqslant b$, defined by the expression $U_A(t,s)=(U_A(s,t))^{-1}$. Thus we have $U_A(t,u)=U_A(t,s)U_A(s,u)$, for any triple $(t,u,s)\varepsilon[0,b]^3$.

2. Boundary value problems for generalized Lyapunov equations.

In order to simplify the statement of the following theorem we introduce the definition of G-function:

<u>Definition 1.</u> Let $A(t)$ be an operator in $L(H)$ for all $t\varepsilon[0,b]$. We say that $A(\cdot)$ is a G-function, if

(i) $\Theta(A(t))\subset\Sigma_{w,\delta}$, $w<0$, $0<\delta<\pi/2$, $t\varepsilon[0,b]$

(ii) There exist constants a and C_A, such that $0<a<1$ and

$$\|(A(t)-A(\tau))A^{-1}(s)\|\leqslant C_A|t-\tau|^a$$

uniformly for all triple $(t,\tau,s)\varepsilon[0,b]^3$.

Under the hypothesis of G-function, the function $A(t)$ generates a fundamental solution $U_A(t,s)$, $0\leq s\leq t\leq b$, for the linear equation $(d/dt)u(t)=A(t)u(t)$, $t\varepsilon[0,b]$, (see lemma 7 of [7]).

Theorem 1. Let us consider the boundary value problem (1.4) where E,F,G, and $A(t)$ are operators in $L(H)$, and let us suppose that $B(\cdot)$ and $C(\cdot)$ are G-functions on $[0,b]$, such that $U_B(t,s)$, $U_C(t,s)$ are fundamental solutions of the systems $(d/dt)u(t)=B(t)u(t)$, and $(d/dt)u(t)=C(t)u(t)$, $t\varepsilon[0,b]$, respectively, with $U_C(t,s)$ invertible in $L(H)$ for $0\leq s\leq t\leq b$. Let M,N and P be operators in $L(H)$ defined by

$$N=EU_B(b,0)\ ;\ P=FU_C(b,0)$$
$$M=-GU_C(b,0)+E\int_0^b U_B(b,s)A(s)U_C(s,0)ds \tag{2.1}$$

Then the boundary value problem has a strong solution, if and only if, the operator equation

$$M+NX-XP=0 \tag{2.2}$$

is solvable. Under this hypothesis, the solution set of problem (1.4) is given by the expression

$$U(t)=U_C(t,0)XU_C(0,t)+\int_0^t U_B(t,s)A(s)U_C(s,t)ds \tag{2.3}$$

where X is a solution of (2.2).

Proof. Let $U(t)$ be a solution of problem (1.4). From the hypothesis there exists $U_C(t,s)$, fundamental solution of $(d/dt)u(t)=C(t)u(t)$, $t\varepsilon[0,b]$. Thus the Cauchy problem $(d/dt)Y(t)=C(t)Y(t)$; $Y(0)=I$, $t\varepsilon[0,b]$, has only one solution $Y(t)$ defined on $[0,b]$. Now let us define $Z(t)=U(t)Y(t)$, then it follows that $(d/dt)Z(t)=(A(t)+B(t)U(t))Y(t)$, for all $t\varepsilon[0,b]$, and $Z(0)=U(0)$. Thus he operator valued function $t\rightarrow\begin{bmatrix}Y(t)\\Z(t)\end{bmatrix}$, satisfies the system

$$(d/dt)\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} C(t) & 0 \\ A(t) & B(t) \end{bmatrix}\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \quad ; \quad \begin{bmatrix} Y(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U(0) \end{bmatrix} \qquad (2.4)$$

From the existence of $U_B(t.s)$ and $U_C(t,s)$, it is easy to show that the operator function

$$W(t,s)= \begin{bmatrix} U_C(t,s) & 0 \\ \int_s^t U_B(t,v)A(v)U_C(v,s)dv & U_B(t,s) \end{bmatrix}$$

is a fundamental solution of (2.4), and from the properties of fundamental solutions, it follows that

$$\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = W(t,0)\begin{bmatrix} I \\ U_o \end{bmatrix} = \begin{bmatrix} U_C(t,0) \\ \int_o^t U_B(t,s)A(s)U_C(s,0)ds + U_B(t,0)U_o \end{bmatrix} \qquad (2.5)$$

where $U_o=U(0)$. As $U(t)$ satisfies the boundary value conditions of (1.4), it follows that

$$EZ(b)=EU(b)Y(b)=(G+U_oF)Y(b) \qquad (2.6)$$

From (2.5)-(2.6) it follows that

$$E\left(\int_o^b U_B(b,s)A(s)U_C(s,0)ds+U_B(b,0)U_o\right)=(G+U_oF)U_C(b,0)$$

$$\left(E \int_o^b U_B(b,s)A(s)U_C(s,0)ds-GU_C(b,0)\right)+EU_B(b,0)U_o-U_oFU_C(b,0)= 0 \qquad (2.7)$$

Thus $U_o=U(0)$ is a solution of (2.1).

Conversely, let us suppose U_o is a solution of the algebraic equation (2.2), where coefficients M,N and P are given by (2.1). Now, let us define define the operator valued functions $Y(t)=U_C(t,0)$ and

$$Z(t)=\int_0^t U_B(t,s)A(s)U_C(s,0)ds+U_B(t,0)U_0$$

From the hypothesis $Y(t)$ is ibertible and $(Y(t))^{-1}=U_C(0,t)$, for all $t\varepsilon[0,b]$. Thus the operator function $X(t)=Z(t)(Y(t))^{-1}$, for $t\varepsilon[0,b]$, is well defined. Computing the derivatives in the strong operator topology, it follows that

$$(d/dt)X(t)=((d/dt)Z(t))\{(Y(t))^{-1}\}-Z(t)\ (Y(t))^{-1}\{(d/dt)Y(t)\}(Y(t))^{-1}= \\ =A(t)+B(t)X(t)-X(t)C(t)$$

with $X(0)=U_0$, and postmultiplying $Z(t)$ by $(Y(t))^{-1}=U_C(0,t)$, and taking into account the property $U_C(s,0)U_C(0,t)=U_C(s,t)$, for $t\varepsilon[0,b]$, and $0\leqslant s\leqslant t\leqslant b$, it follows that $X(t)$ is given by

$$X(t)=U_C(t,0)U_0U_C(0,t)+\int_0^t U_B(t,s)A(s)U_C(s,t)ds$$

Thus the result is proved.

Theorem 1 allows us to obtain explicit solutions of the boundary value problem (1.4) in terms of solutions of the algebraic Lyapunov equation (2.2) whith coefficients defined by (2.1). For particular classes of operators M,N and P, a complete description of solutions of equaions of type (2.2) may be found in [1], [4]. In the following corollary conditions on coefficients of (2.2) in order to ensure the existence and uniqueness for solutions of (1.4) are presented. Also, conditions for obtaining an explicit expression of solutions are given.

Corollary 1. Let us consider the notation of th. 1, let $\sigma_\delta(N)=\{z\varepsilon C; zI-N$ is not onto$\}$ the approximate defect spectrum of N, and let $\sigma_\pi(P)=\{z\varepsilon C;\ zI-P$ is not bounded below$\}$ the approximate point spectrum of P. Then

(i) Problem (1.4) is solvable if

$$\sigma_\delta(N)\ \cap\ \sigma_\pi(P)=\emptyset \qquad (2.8)$$

(ii) Problem (1.4) has only one solutions if

$$\sigma(N) \cap \sigma(P)=\emptyset \qquad (2.9)$$

(iii) If N is an algebraic operator annihilated by the polynomial $p(z)=\sum_{k=o}^{n} a_k z^k$, under the hypothesis (3.1), the only solution U_o of equation (2.2) is given by

$$U_o=-GZ^{-1}, \quad G=-\sum_{k=1}^{n}\sum_{j=1}^{k} a_k N^{k-j} M P^{j-1}; Z=p(P) \qquad (2.10)$$

In both cases for a solution U_o of (2.2), the expression (2.3) with $X=U_o$ provides an explicit expression of solutions of (1.4)

Proof. (i) From th.5, of [3], and from (2.8), equation (2.2) is solvable. (ii) It is a consequence of the hypothesis and Rosenblun's theorem, [12], p.8. The part (iii) is a consequence of corollary 1 of [5].

Next result is concerned with the problem of finding b-periodic solutions of the equation

$$(d/dt)U(t)=A(t)+B(t)U(t)-U(t)C(t) \qquad (2.11)$$

Corollary 2. Let us consider equation (2.11) where coefficient functions $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$, satisfy the hypothesis of th. 1, and are b-periodic, $b>0$ on the real line. Let N, M and P be operators in L(H) defined by the expressions

$$N=U_B(b,0); P=U_C(b,0); \quad M=\int_o^b U_B(b,s)A(s)U_C(s,0)ds \qquad (2.12)$$

Then there exist b-periodic solutions of (2.11), if and only, equation (2.2) with coefficients given by (2.12) is solvable. In this case the solution set of b-periodic solutions of (2.11) is given by (2.3) where X is a solution of (2.2), (2.12).

Proof. Let us consider the boundary value problem (1.4) with E=F=I and G=0. Then from th. 1, there exist solutions of (2.11) with U(b)=U(0), if and only if, equation (2.2),(2.12) is solvable. Given a solution U(t) of equation (2.11) is clear that one gets a b-periodic solution of (2.11) on the real line, by extending the solution b-periodically on all the real line. The result is now a consequence of theorem 1.

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References.

[1] C. APOSTOL, On the operator equation TX-XV=A, Proc. Amer. Math. Soc. Vol. 59, N. 1 (1976), 115-118.

[2] R.S. BUCY and P.D. JOSEPH, Filtering for stochastic procceses witl Applications to Guidance, Interscience, New York, 1968.

[3] C. DAVIS and P. ROSENTHAL, Solving linear equations, Can. J. Math. Vol. XXVI, No. 6 (1974), 1384-1389.

[4] J.A. GOLDSTEIN, On the operator equation AX+XB=Q, Proc. Amer. Math. Soc. Vol. 70. No. 1 (1978).

[5] V. HERNANDEZ and L. JODAR, Sobre la ecuación cuadrática en operadores A+BT+TC+TDT=0, Stochastica Vol. VII, Nº 2 (1983), 145-154.

[6] D.A. HERRERO, Approximation of operators in Hilbert space, Pitman Pub. Co., Research Notes in Maths. Vol. 72, London , 1982.

[7] H.J. KUIPER, Generalized Operator Riccati Equations, Technical Report Nº 69 Arizona State University, April 1982, (To appear in SIAM. J. Math. Anal.).

[8] W. H. KWON and A. E. PEARSON, Linear Systems with Two-Point Boundary Lyapunov and Riccati Equations, IEEE Trans. on Autom. Control, AC-27 (1982), 436-441.

[9] G.E. LADAS and V. LAKSHMIKANTHAM, Differential Equations in Abstract Spaces, Academic Press, New York, 1972.

[10] J.L. LIONS, Optimal Control of Systems Governed by Partial Differential Equations, Springer Verlag, New York, 1970.

[11] L. JÓDAR, Boundary Problems for Riccati and Lyapunov Equations, Proc. Edinburgh Mat. Soc. 29 (1986), 15-21.

[12] H. RADJAVI and P. ROSENTHAL, Invariant subspaces, Springer Verlag, Berlin, 1973.

[13] R. REDHEFFER, On the relacion of transmission line theory to scattering and transfers, J. Math. Physics, 41 (1962), 1-41.

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