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ON INDEPENDENCE IN SOME FAMILIES OF MULTIVARIATE DISTRIBUTIONS

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ABSTRACT

In this paper we will prove a characterization for the independence of random vectors with positive (negative) orthant dependence according to a direction. The result can be seen as a generalization of a result by Lehmann [4].

1. Introduction.

If two random variables X and Y are independent, then X and Y are uncorrelated, that is, cov(X,Y)=0. The reverse is not true in general. If the distribution is the bivariate normal, then uncorrelation also implies independence. We determined to study some conditions for the variables, under which the knowledge of uncorrelation of the variables would be enough to assure independence. These conditions will determine a family of bivariate distributions where independence will be characterized by the vanishing of a measure of association for the variables (in this case the covariance).

This problem was solved by Lehmann [4]. He introduced the family of bivariate distributions that are quadrant dependent. He proved, by using an identity from Hoeffding [3], that, in this family, independence is equivalent to uncorrelation. We want to find conditions such that combined with the "uncorrelation" of the variables (that we have to specify), they assure us of the independence of these in the multivariate case; that is, we will try to generalize Lehmann's result for n random variables.

The family determined by these conditions is constituted by all the multivariate distributions with positive or negative orthant dependence according to a direction. This concept of dependence was introduced by Quesada [5], and is a generalization of the orthant dependence studied by Esary, Proschan and Walkup [1]. The "uncorrelation" of the variables, which we will call "mutual uncorrelation", will not only express uncorrelation of all the variables pairwise, because independence pairwise of the variables is not enough to warranty independence.

In section 2, we will recall the concepts and results for the bivariate case and in section 3 we will prove the generalizations for the multivariate case.

2. Bivariate dependence.

Let X and Y be two random variables, and let F, F_X and F_Y be the joint and marginal distribution functions respectively.

Definition 2.1. X and Y are positively quadrant dependent if

$$F(x,y) \geqslant F_X(x)\cdot F_Y(y) \quad \text{for every} \quad x,y \in R.$$

Similarly, the negative quadrant dependence is obtained by changing the inequality sign.

The following lemma from Hoeffding [2] is essential.

Lemma 2.2. (Hoeffding). If E(XY), EX and EY exist, then

$$E(XY)-EX\cdot EY = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (F(x,y)-F_X(x)\cdot F_Y(y))\,dxdy$$

By using this lemma, Lehmann proved the following result:

Theorem 2.3. (Lehmann). If X and Y are two random variables with positive quadrant dependence, and E(XY), EX and EY exist, then

$$E(XY) \geqslant EX\cdot EY$$

and the equality holds if and only if X and Y are independent.

This result establishes the equivalence between independence and uncorrelation for random variables with positive quadrant dependence (PQD). An analogous result is obtained for random variables with negative quadrant dependence (NQD) by changing the inequality sign. So, if X and Y are uncorrelated and are PQD or NQD, then they are independent.

3. Multivariate dependence

Quadrant dependence has been generalized for n random variables by Esary, Proschan and Walkup [1]. In [5], Quesada considered a generalization of quadrant dependence which includes the concept of Esary, Proschan and Walkup.

Definition 3.1. Let $X_1, X_2, \ldots, X_n$ be n random variables and let $\underset{\sim}{\alpha} \varepsilon R^n$ such that $|\alpha_i|=1$, $i=1,2,\ldots,n$. $X_1, X_2, \ldots, X_n$ are positively orthant dependent according to the direction $\underset{\sim}{\alpha} \varepsilon R^n$ (POD($\underset{\sim}{\alpha}$)) if

$$P\{\bigcap_{i=1}^{n} (\alpha_i X_i \geqslant x_i)\} \geqslant \prod_{i=1}^{n} P\{\alpha_i X_i \geqslant x_i\} \tag{3.1}$$

for every $\underset{\sim}{x}=(x_1,x_2,\ldots,x_n)\in R^n$.

Similarly, $X_1,X_2,\ldots,X_n$ are negatively orthant dependent according to α (NOD($\underset{\sim}{\alpha}$)) if (3.1) is verified with the reverse inequality.

If $X_1,X_2,\ldots,X_n$ are positively orthant dependent according to $\underset{\sim}{\alpha}\in R^n$, then big values of X_i for $i\in J$ are associated with small values of X_i for $i \in I-J$, where $I=\{1,2,\ldots,n\}$ and $J=\{i\in I \ / \ \alpha_i=1\}$. In the case of negative orthant dependence according to a direction NOD($\underset{\sim}{\alpha}$), there is no association between big values of $\{X_i,\ i \in J\}$ and small values of $\{X_i,\ i \in I-J\}$.

For n=2 we obtain POD(1,1) $\Leftrightarrow$ NOD(-1,1) $\Leftrightarrow$ NOD(1,-1) $\Leftrightarrow$ $\Leftrightarrow$ POD(-1,-1) $\Leftrightarrow$ PQD and NOD(1,1) $\Leftrightarrow$ NOD(-1,-1) $\Leftrightarrow$ POD(-1,1) $\Leftrightarrow$ POD(1,-1) $\Leftrightarrow$ NQD.

For $\underset{\sim}{\alpha}=(-1,-1,\ldots,-1)$, we have the model of association defined by Esary, Proschan and Walkup, because if $\underset{\sim}{X}$ is POD(-1,-1,...,-1), then

$$P\{\bigcap_{i=1}^{n}(X_i<x_i)\} \geqslant \prod_{i=1}^{n} P\{X_i< x_i\}$$

for every $\underset{\sim}{x}=(x_1,x_2,\ldots,x_n)\in R^n$. It is the same for negative dependence.

Now we will define what we call "mutual uncorrelation" for n random variables as a generalization of the bivariate uncorrelation.

<u>Definition 3.2.</u> The random variables $X_1,X_2,\ldots,X_n$ are said to be "mutually uncorrelated" if

$$\mathrm{cov}(\prod_{i\in J_1} X_i, \prod_{i\in J_2} X_i) = 0$$

for every $J_1,J_2 \subset I$ such that $J_1\cap J_2=\emptyset$.

For $n=2$, the "mutual uncorrelation" is the known bivariate uncorrelation.

Our main result can now be enunciated, although we will prove it later.

Theorem 3.3. If $X_1, X_2, \dots, X_n$ are random variables positively or negatively orthant dependent according to a direction $\underset{\sim}{\alpha} \in R^n$, then the independence of $X_1, X_2, \dots, X_n$ is equivalent to the "mutual uncorrelation" of these.

For the proof of theorem 3.3, we will first prove some other results.

Lemma 3.4. The random variables $X_1, X_2, \dots, X_n$ are "mutually uncorrelated" if and only if

$$E\Big(\prod_{j\in J} X_j\Big) = \prod_{j\in J} EX_j \qquad \text{for every subset} \quad J\subset I. \tag{3.2}$$

Proof. If $X_1, X_2, \dots, X_n$ are "mutually uncorrelated" and $J\subset I$, then for $J_1=J-\{1\}$ and $J_2=\{1\}$, we obtain

$$E\Big(\prod_{j\in J} X_j\Big) = EX_1 \cdot E\Big(\prod_{j\in J_1} X_j\Big),$$

and with this reasoning we have the result. If (3.2) is satisfied, and $J_1, J_2 \subset I$ such that $J_1 \cap J_2 = \emptyset$, then by applying (3.2) to J_1, J_2 and $J_1 \cup J_2$, we obtain the "mutual uncorrelation".

Theorem 3.5. Let $X_1, X_2, \dots, X_n$ be random variables such that any (n-1) of them are "mutually uncorrelated", and let $Y_1, Y_2, \dots, Y_n$ be random variables independent of the preceding, and with the same distribution of these if n is even, or with the distribution of $(-X_1, X_2, \dots, X_n)$ if n is odd. Then

$$E\Big(\prod_{i\in I} (X_i - Y_i)\Big) = 2\Big(E\Big(\prod_{i\in I} X_i\Big) - \prod_{i\in I} E\, X_i\Big)$$

Proof. If n is odd, then with the conditions of the theorem we can obtain that

$$E(\prod_{i\in I}(X_i-Y_i))=2\cdot\{E(\prod_{i\in I}X_i)+\sum_{J\in R}(-1)^{Card(J)}E(\prod_{j\in J}X_j)E(\prod_{j\in I-J}X_j)-$$

$$-\sum_{J\in P}(-1)^{Card(J)}E(\prod_{j\in J}X_j)E(\prod_{j\in I-J}X_j)\}$$

where R and P are given by

$$R=\{J : J\subset I,\quad 1\notin J,\quad 1\leqslant Card(J)<\tfrac{n}{2}\}$$

$$P=\{J : J\subset I,\quad 1\in J,\quad Card(J)<\tfrac{n}{2}\}$$

By using the fact that any (n-1) of $X_1,X_2,\ldots,X_n$ are "mutually uncorrelated" and lemma 3.4, we obtain the result.

If n is even, then we obtain

$$E(\prod_{i\in I}(X_i-Y_i))=2\cdot\{E(\prod_{i\in I}X_i)+\sum_{J\in H}(-1)^{Card(J)}E(\prod_{j\in J}X_j)E(\prod_{j\in I-J}X_j)\}+$$

$$+\sum_{J\in L}(-1)^{n/2}E(\prod_{j\in J}X_j)E(\prod_{j\in I-J}X_j)$$

where H and L are given by

$$H=\{J : J\subset I,\quad 1\leqslant Card(J)<\tfrac{n}{2}\}$$

$$L=\{J : J\subset I,\quad Card(J)=\tfrac{n}{2}\}$$

With the same reasoning as before, the proof is concluded.

Theorem 3.6. Let $X_1,X_2,\ldots,X_n$ be random variables such that any (n-1) of them are independent and let $Y_1,Y_2,\ldots,Y_n$ be random variables independent of the preceding and with the same distribution of these if n is even, or with the distribution of

$(-X_1, X_2, \dots, X_n)$ if n is odd. Then if n is even,

$$E\{\prod_{i\in I}(I(u_i,X_i)-I(u_i,Y_i))\}=2\cdot\{P(\bigcap_{i\in I}A_i)-\prod_{i\in I}P(A_i)\}$$

and if n is odd,

$$E\{\prod_{i\in I}(I(u_i,X_i)-I(u_i,Y_i))\}=\{P(B_1\bigcap_{i=2}^{n}A_i)-P(B_1)\cdot\prod_{i=2}^{n}P(A_i)\}-$$

$$-\{P(\bigcap_{i\in I}A_i)-\prod_{i\in I}P(A_i)\},$$

where $A_i=\{\omega: X_i(\omega)>u_i\}$, $i=1,2,\dots,n$ and $B_1=\{\omega: -X_1(\omega)>u_1\}$,

and

$$I(u,x)=\begin{cases}1 & \text{if } x>u\\ 0 & \text{if } x\leqslant u\end{cases}$$

<u>Proof.</u> Let H, L, R and P be the same as in theorem 3.5. If n is even, we have that

$$E\{\prod_{i\in I}(I(u_i,X_i)-I(u_i,Y_i))\}=2\cdot\{P(\bigcap_{i\in I}A_i)+\sum_{j\in H}(-1)^{\mathrm{Card}(J)}P(\bigcap_{j\in J}A_j)P(\bigcap_{j\in I-J}A_j)\}+$$

$$+\sum_{j\in L}(-1)^{n/2}P(\bigcap_{j\in J}A_j)P(\bigcap_{j\in I-J}A_j)$$

If n is odd, we obtain that

$$E\{\prod_{i\in I}(I(u_i,X_i)-I(u_i,Y_i))\}=\{P(B_1\bigcap_{i=2}^{n}A_i)+$$

$$+\sum_{J\in R}(-1)^{\mathrm{Card}(J)}P(\bigcap_{j\in J}A_j)P(B_1\bigcap_{j\in I_1-J}A_j)-\sum_{J\in P}(-1)^{\mathrm{Card}(J)}P(B_1\bigcap_{j\in J_1}A_j)P(\bigcap_{j\in I-J}A_j)\}-$$

$$-\{P(\bigcap_{i\in I}A_i)+\sum_{J\in R}(-1)^{\mathrm{Card}(J)}P(\bigcap_{j\in J}A_j)P(\bigcap_{j\in I-J}A_j)-$$

$$-\sum_{J\in P}(-1)^{\mathrm{Card}(J)}P(\bigcap_{j\in J}A_j)P(\bigcap_{j\in I-J}A_j)\}$$

where $J_1=J-\{1\}$ and $I_1=I-\{1\}$.

By using the fact that any (n-1) of $X_1,X_2,\ldots,X_n$ are independent, we obtain the result.

Now, we can prove theorem 3.3.,

Proof theorem 3.3. It is clear that independence implies "mutual uncorrelation". The reverse implication will be proven by induction on n. If n=2, the "mutual uncorrelation" is reduced to the usual bivariate uncorrelation and our result is reduced to Lehmann's result. Let us assume that the result is satisfied for (n-1) and let us prove it for n. The "mutual uncorrelation" of $X_1,X_2,\ldots,X_n$ implies that any (n-1) of them are also "mutually uncorrelated". Moreover, because $X_1,X_2,\ldots,X_n$ are POD($\underset{\sim}{\alpha}$) or NOD($\underset{\sim}{\alpha}$), then any (n-1) of them are also POD($\underset{\sim}{\alpha}^*$) or NOD($\underset{\sim}{\alpha}^*$), where $\underset{\sim}{\alpha}^*$ is the resulting vector after eliminating in $\underset{\sim}{\alpha}$ the component corresponding to the variable that is excluded.

Then, by inductive hypothesis, any (n-1) variables between $X_1,X_2,\ldots,X_n$ are independent, and $X_1,X_2,\ldots,X_n$ satisfy the conditions in theorem 3.5 and 3.6. Therefore, $\alpha_1X_1,\alpha_2X_2,\ldots,\alpha_nX_n$ also satisfy these conditions, and so, if $Y_1,Y_2,\ldots,Y_n$ have the same distribution as $X_1,X_2,\ldots,X_n$ when n is even, or the distribution of $-X_1,X_2,\ldots,X_n$ when n is odd, then

$$(3.3)\quad E\left(\prod_{i\in I}(\alpha_iX_i-\alpha_iY_i)\right)=E\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left\{\prod_{i\in I}(I(u_i,\alpha_iX_i)-I(u_i,\alpha_iY_i))\right\}du_1du_2\ldots du_n$$

Because we are assuming that all the moments $E(\prod_{j\in J}\alpha_jX_j)$ exist for every $J\subseteq I$, then we can compute the expectation under the integral signs. As $X_1,X_2,\ldots,X_n$ are "mutually uncorrelated", and by using theorem 3.5, the left side in (3.1) is zero for either n even or odd. By theorem 3.6.

$$E\left\{\prod_{i\in I}(I(u_i,\alpha_iX_i)-I(u_i,\alpha_iY_i))\right\}\quad\text{is non-negative}$$

if either $\underset{\sim}{X}$ is POD($\underset{\sim}{\alpha}$) and n is even or $\underset{\sim}{X}$ is NOD($\underset{\sim}{\alpha}$) and n is odd, and non-positive in the remaining cases. Therefore, in any case, we obtain that

$$P(\bigcap_{i\varepsilon I} A_i) = \prod_{i\varepsilon I} P(A_i), \qquad \text{or}$$

$$P\{\alpha_1 X_1 > u_1, \alpha_2 X_2 > u_2, \ldots, \alpha_n X_n > u_n\} = \prod_{i=1}^{n} P\{\alpha_i X_i > u_i\}$$

for every $(u_1, u_2, \ldots, u_n)\varepsilon R^n$ except, perhaps, on a set with zero Lebesgue's measure. By using the right-continuity of $P\{\bigcap_{i\varepsilon I}(\alpha_i X_i > u_i)\}$, (3.4) is satisfied for every $(u_1, u_2, \ldots, u_n)\varepsilon R^n$ and therefore $\alpha_1 X_1, \alpha_2 X_2, \ldots, \alpha_n X_n$ are independent, and so are $X_1, X_2, \ldots, X_n$.

Remarks. It is necessary to point out that (3.4) is obtained for n odd as well as for n even. This is so because, for instance, if the random variables are POD($\underset{\sim}{\alpha}$), then

$$P(B_1 \bigcap_{i=2}^{n} A_i) \leqslant P(B_1) \cdot \prod_{i=2}^{n} P(A_i) \qquad \text{and}$$

$$P(\bigcap_{i\varepsilon I} A_i) \geqslant \prod_{i\varepsilon I} P(A_i) \quad ,$$

and (3.4) is obtained. It is similar for the case of NOD($\underset{\sim}{\alpha}$).

I should mention that the main result in this paper has been independently obtained later by Fang [2], by using a different technique.

References.

[1] ESARY, J., PROSCHAN, F. and WALJUP, W. Association of random variables with applications. Ann. Math. Statist. 38 (1967),

1466-1474.

[2] FANG, Z. Dependence classes and independence. Ph. D. dissertation (1986), University of Pittsburgh.

[3] HOEFFDING, W. Masstabinvariante Korrelations-theorie. Schriften Math. Inst. Univ. Berlin 5 (1940), 181-233.

[4] LEHMANN, E. L. Some concepts of dependence. Ann. Math. Statist. 37 (1966), 1137-1153.

[5] QUESADA, J. J. Conceptos de dependencia, medidas de asociación y aplicaciones. Tesis doctoral (1985), Univ. de Granada, Spain.

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