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SOME REMARKS ON A PROBLEM OF C. ALSINA

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SUMMARY

Equation

$$(1)\quad f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(f(x)+y))$$

has been proposed by C. Alsina in the class of continuous and decreasing involutions of $(0,+\infty)$. *General solution of (1) is not known yet. Nevertheless we give solutions of the following equations which may be derived from (1):*

$$(2)\qquad f(x+1) + f(f(x)+1) = 1,$$

$$(3)\qquad f(2x) + f(2f(x)) = f(2f(x+f(x))).$$

Equation (3) leads to a Cauchy functional equation

$$(4)\qquad \varphi(f(x)+x) = \varphi(f(x)) + \varphi(x),$$

restricted to the graph of the function f, of the type not yet considered. We describe a general solution as well as we give some conditions sufficient for the uniqueness of solutions of (2) and (4).

Key words: functional equation, continuous solution, characterization of the inverse.

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C. Alsina proposed at the 24th International Symposium on Functional Equations to solve the following equation (cf. [1])

$$(1)\qquad f(x+y)+f[f(x)+f(y)] = f[f(x+f(y))+f(y+f(x))]$$

where f is a continuous and decreasing involution of $(0,+\infty)$. It is easy to see that f given by $f(x)=ax^{-1}$ for a positive a fulfils (1) and all the requirements. However it is not known whether there any other solutions.

Without loss of generality we may restrict ourselves to the case where 1 is the only fixed point of f (cf. Z. Moszner [10] and T. M. K Davison [4]). It turns out that any solution of (1) fulfilling $f(1)=1$ satisfies

$$(2)\qquad f(x+1)+f(f(x)+1)=1$$

for $x \in (0,+\infty)$ (cf. Davison [4], Sablik [11]). It may be interesting that equation (2) has already been dealt with. W. Benz and S. Elliger in their paper [3] proved that the inverse is the only endomorphism (or antiendomorphism) of the multiplicative group K^* of a field K which fulfils (2). The same equation appears also in the definition of the so called KT-nearfields (cf. R. Artzy [2]).

Another equation which may be derived from (1) after substituting x in the place of y is

$$(3)\qquad f(2x)+f(2f(x))=f(2f(x+f(x))).$$

As we show later, (3) leads to a Cauchy equation on the graph of f but of the type that has not been studied yet (note that the number of results concerning this topic is quickly increasing, cf. e.g. M.C. Zdun [13], G.L. Forti [6], J. Dhombres [5], J. Matkowski [9] or M. Sablik [12]).

We give general solutions of (2) and (3) in classes of functions slightly more general than the one proposed by C. Alsina. Then we give also some conditions assuring that the inverse is the only solution of (2) or (3).

1. In the present section we will describe solutions of (2). First we prove some lemmas which enumerate properties of solutions of (2). Let us start with the following.

Lemma 1. Let $f:(0,+\infty)\to(0,+\infty)$ be a solution of (2) which is invertible in $(1,+\infty)$. Then f is an involution of $(0,+\infty)$, i.e. $f^2=id$. In particular f is a one-to-one mapping of $(0,+\infty)$ onto itself.

Proof. Fix arbitrarily $x\in(0,+\infty)$. Putting f(x) instead of x into (2) we obtain

$$f(f(x)+1)+f(f(f(x))+1)=1. \tag{4}$$

Comparing left hand sides of (2) and (4) we see that

$$f(x+1)=f(f(f(x))+1).$$

Hence $x=f(f(x))$ follows, since both $x+1$ and $f(f(x))+1$ are in $(1,+\infty)$, where f is invertible.

Remark. Invertibility assumption in the above lemma is essential as the example of $f:(0,+\infty)\to(0,+\infty)$ set arbitrary for x in $(0,1]$ and equal to 1/2 in $(1,+\ )$ testifies.

Lemma 2. If $f:(0,+\infty)\to(0,+\infty)$ is a solution of (2) and $f(1)=1$ then $f(2)=1/2$. If, moreover, f is invertible in $(1,+\infty)$ then $f(x)=2-x$ if and only if $x=1$.

Proof. First statement is obvious and to prove the second one suppose that $f(1+c)=2-(1+c)=1-c$ for a $c>0$. Then (2) implies

$$f(f(c)+1)=1-f(c+1)=c.$$

But by Lemma 1 f is an involution so we infer

$$f(c)+1=f(c)$$

which is a contradiction. This ends the proof for if $f(1-c)=2-(1-c) =1+c$ for a $c>0$ then using Lemma 1 again we get $f(1+c)=1-c$ which has already been proved to be impossible.

Now let us enumerate some properties of continuous solutions of (2) which are invertible in $(1,+\infty)$.

Lemma 3. Let $f:(0,+\infty)\to(0,+\infty)$ be a continuous solution of (2) which is invertible in $(1,+\infty)$. Then f is decreasing, $\lim_{x\to 0+} f(x)=+\infty$, $\lim_{x\to+\infty} f(x)=0$ and 1 is the only fixed point of f.

Proof. By Lemma 1, f is a continuous involution of $(0,+\infty)$. It is easy to see that it is the only continuous and increasing involution. But it does not solve (2) and therefore f has to be decreasing. Hence the existence and values of limits of f at 0 and $+\infty$ as well as the uniqueness of a fixed point. Letting x tend to $+\infty$ in (2) and using continuity of f we find out that $f(1)=1$.

Summarizing the above results we have

Proposition 1. If $f:(0,+\infty)\to(0,+\infty)$ is a continuos solution of (2) which is invertible in $(1,+\infty)$ then $f_1:=f|_{[1,2]}$ is continuous, decreasing, $f_1(1)=1$, $f_1(2)=1/2$ and $f_1(x)\neq 2-x$ for $x\in(1,2]$.

The main result of this section is the following reverse of Proposition 1.

Theorem 1. Let $f_1:[1,2]\to R$ be any continuous and decreasing function fulfilling $f_1(1)=1$, $f_1(2)=1/2$ and $f_1(x)\neq 2-x$ for $x\in(1,2]$. Then f_1 can be uniquely extended to asolution $f:(0,+\infty)\to(0,+\infty)$ of (2). This solution is a continuous involution of $(0,+\infty)$.

The above theorem will follow in an obvious way from a slightly more general result we prove below.

Proposition 2. Let $f_1:[1,2]\to R$ be an arbitrary continuous and decreasing function which fulfils $f_1(1)=1$ and $f_1(2)=1/2$. Put $c_o:=\sup\{c\ [0,1]:f_1(1+c)=1-c\}$. Then f_1 can uniquely be extended to a function $f:(c_o,+\infty)\to(c_o,+\infty)$ which solves (2) in $(c_o,+\infty)$. This function is a continuous involution of $(c_o,+\infty)$.

Proof. Let f_1 be a function fulfilling admitted assumptions. Observe that $f_1([1,2])=[1/2,1]\subset(0,1]$. Denote $I_1=[1,2]$, and $I_n=(n,n+1]$ for $n\geqslant 2$. We can define a sequence $(f_n)_{n\in N}$ of mappings $f_n:I_n\to(0,1]$ taking f_1 as above and setting

$$f_{n+1}(x)=1-f_1(f_n(x-1)+1),\quad x\in I_{n+1}, \tag{5}$$

for all $n\in N$. To make sure that functions f_n, $n\geqslant 2$, may be defined by (5) we use induction. Since $f_1(I_1)\subset(0,1]$, f_2 is well defined (notice that if $x\in I_{n+1}$ then $x-1\in I_n$, $n\in N$, and $f_1(x-1)+1\in(1,2]$ for every $x\in I_2$). Moreover, $f_2(I_2)\subset(0,1]$ as it may be easily checked. Now, if $f_n:I_n\to(0,1]$ is defined, then $f_n(x-1)+1\in(1,2]$ for every $x\in I_{n+1}$ and therefore f_{n+1} is well defined by (5) and takes values in $(0,1]$, which follows from the properties of f_1.

An easy induction shows also that f_n are continuous and decreasing for $n\in N$. Moreover, we have for every $n\in N$

$$\lim_{x\to(n+1)+} f_{n+1}(x)=f_n(n+1). \tag{6}$$

Indeed, for n=1 we have in virtue of the continuity of f_1

$$\lim_{x\to 2+} f_2(x)=\lim_{x\to 2+}(1-f_1(f_1(x-1)+1))=1-f_1(f_1(1)+1)=1-(1/2)=1/2=f_1(2)$$

If (6) holds for an $n\in N$ then it holds for $n+1$, by continuity of f_1:

$$\lim_{x\to(n+2)+} f_{n+2}(x) = \lim_{x\to(n+2)+} (1-f_1(f_{n+1}(x-1)+1)$$

$$=1-f_1(\lim_{x\to(n+2)+} f_{n+1}(x-1)+1) = 1-f_1(f_n(n+1)+1)$$

$$=f_{n+1}(n+2),$$

which ends the induction.

Define $f_+:[1,+\infty)\to(0,1]$ putting $f_+(x):=f_n(x)$, whenever $x\in I_n$. Taking into account our previous observations we infer that f_+ is continuous and decreasing. The latter property implies that

(7) $$d=\lim_{x\to+\infty} f_+(x)$$

exists and is in $[0,1]$. Moreover, f_+ fulfils (2) for every $x\in[1,+\infty)$. Indeed, let $x\in I_n$ for some $n\in\mathbb{N}$. Then $f_+(x)=f_n(x)$, $f_+(x+1)=f_{n+1}(x+1)$ and $f_+(x)+1\in(1,2]$. Thus

$$f_+(x+1)+f_+(f_+(x)+1)=f_{n+1}(x+1)+f_1(f_n(x)+1)=1$$

by (5) which proves the required equality. Letting x tend to $+\infty$ in (2) we obtain in view of (7) and continuity of f_+

$$f_+(d+1)=1-d$$

whence $d\in[0,c_o]$. If $c_o=0$ then $d=c_o=0$. Suppose that $0\leqslant d<c_o\leqslant 1$. Then continuity of f_+ makes possible choice of an $x_o\in[1,+\infty)$ such that $f_+(x_o)=c_o$. Using (2), definition of c_o and continuity of f_1 which imply $f_+(c_o+1)=1-c_o$, we obtain

$$f_+(x_o+1)=c_o.$$

An easy induction shows that $f_+(x_o+n)=c_o$ for $n\in\mathbb{N}$. This leads to a contradiction with (7) and therefore $d=c_o$. Thus, by monotonicity of f_+ we obtain $f_+([1,+\infty))=(c_o,1]$. As f_+ is invertible we are able to define $f:(c_o,+\infty) \to (c_o,+\infty)$ by

$$f(x)=\begin{cases} f_+(x) & \text{for } x\in[1,+\infty), \\ f_+^{-1}(x) & \text{for } x\in(c_o,1). \end{cases}$$

Of course f is a continuous involution. We have already observed that f fulfils (2) for $x\in[1,+\infty)$. If $x\in(c_o,1)$ then $f(x)=f_+^{-1}(x)1,+\infty)$ and hence

$$\begin{aligned} f(x+1)+f(f(x)+1)&=f(f(f(x))+1)+f(f(x)+1) \\ &=f_+(f_+(f(x))+1)+f_+(f(x)+1)=1. \end{aligned}$$

It is easy to check that for any solution $\bar{f}:(c_o,+\infty)\to(c_o,+\infty)$ of (2) such that $\bar{f}|_{I_1}=f_1$ we have $\bar{f}|_{I_n}=f_n$, $n\in N$, where f_n are given by (5). Hence $\bar{f}|_{[1,+\infty)}=f|_{[1,+\infty)}$ and finally if $x\in(c_o,1)$ then

$$f(\bar{f}(x)+1)=\bar{f}(\bar{f}(x)+1)=1-\bar{f}(x+1)=1-f(x+1)=f(f(x)+1).$$

Thus $\bar{f}(x)=f(x)$ by invertibility of f. This ends the proof.

To get Theorem 1 it is enough to observe that $c_o=0$ under its assumptions.

<u>Remark.</u> Let us observe that if f_1 is of higher regularity (C^r, $1\leqslant r\leqslant+\infty$) then it may be extended to a solution of (2) which is of the same regularity. It is enough to impose some natural boundary conditions on f_1 and apply a similar proof as in Proposition 2.

Now we are going to give two conditions which characterize the inverse among solutions of (2).

<u>Theorem 2.</u> Let $f:(0,+\infty)\to(0,+\infty)$ be a solution of (2) which is invertible in $(1,+\infty)$, $f(1)=1$, and suppose that $g:[1/2,2]\to R$ gi-

ven by $g(x)=xf(x)$ is monotonic either in $[1/2,1]$ or in $[1,2]$. Then $f(x)=x^{-1}$ for every $x\in(0,+\infty)$.

<u>Proof.</u> By Lemma 2 we get $f(2)=1/2$, and since by Lemma 1 f is an involution we have also $f(1/2)=2$. Thus $g(1/2)=g(1)=g(2)=1$ and monotonicity of g implies $g(x)=1$ in $[1/2,1]$ or in $[1,2]$. In both cases however $f(x)=x^{-1}$ for $x\in[1/2,2]$ because f is an involution. The function $f_1=f|_{[1,2]}$ fulfils the assumptions of Theorem 1 and therefore it has a unique extension to a solution of (2). Since the inverse actually is such an extension we obtain our assertion.

The final result of this section reads as follows.

<u>Theorem 3.</u> Let $f:(0,+\infty)\to(0,+\infty)$ be a solution of (2) which is invertible in $(1,+\infty)$ and $f(1)=1$. If $g=1/f$ is convex or concave in $[1/2,2]$ then $f(x)=x^{-1}$ for every $x\in(0,+\infty)$.

<u>Proof.</u> Lemmas 1 and 2 imply $f(2)=1/2$ and $f(1/2)=2$. Thus $g(x)=x$ for $x\in\{1/2,1,2\}$. It is easy to see that both convexity or concavity of g imply $g(x)=x$ for $x\in[1/2,2]$. In particular $f_1(x):=f|_{[1,2]}(x)$ $=x^{-1}$ for $x\in[1,2]$ and using Theorem 1 we get $f(x)=x^{-1}$ for all $x\in(0,+\infty)$.

<u>Remarks.</u> 1. It is clear that any solution $f:(0,+\infty)\to(0,+\infty)$ of (2) is bounded above by 1 in $(1,+\infty)$. Therefore it is enough to assume in Theorem 2 that g is Jensen (mid-point) concave. Indeed, then g is bounded below in $(1,2)$ and hence it has to be continuous by well known results (cf. e.g. M. Kuczma [8]).

2. $f:(0,+\infty)\to(0,+\infty)$ given by

$$f(x)=\begin{cases} 2^{-n}x+(n+2)2^{-n} & \text{for } x\in[n,n+1) \\ -2^{n}x+n+2 & \text{for } x\in[2^{-n},2^{-n+1}) \end{cases},\quad n\in\mathbb{N}$$

is a convex and continuous involution solving (2). This shows that convexity does not characterize the inverse among solutions of (2).

2. Now let us focus our attention on solving equation

(3). In Alsina's problem f is an involution (this actually follows from the remaining assumptions, cf. Moszner [10]). Thus, setting $\varphi(x):=f(2f(x))$ for $x \in (0,+\infty)$ we can rewrite (3) in equivalent form

$$\varphi(x+f(x))=\varphi(x)+\varphi(f(x)). \tag{8}$$

The above equation means that φ is additive for all pairs $(x,f(x))$, $x \in (0,+\infty)$, i.e. φ is additive on the graph of f. As we have mentioned in the introduction additivity on graphs has been widely investigated. However, usually point (0,0) was assumed to be an accumulation point of the given graph, which is not the case in our present situation (it may easily be observed that $\lim_{x\to 0+} f(x)=+\infty$).

First result of this section describes a general solution of (8) as "depending on an arbitrary function" (cf. M. Kuczma [7] for an explanation of this notion).

<u>Theorem 4.</u> Let $f:(0,+\infty)\to(0,+\infty)$ be an involution satisfying $f(1)=1$ and $f((0,1))\subset(1,+\infty)$ and $f((1,+\infty))\subset(0,1)$. Then every function $\varphi_o:[1,+\infty)\to R$ such that

$$\varphi_o(2)=2\varphi_o(1) \tag{9}$$

can uniquely be extended to a solution $\varphi:(0,+\infty)\to R$ of the equation (8). Moreover if f and φ_o are continuous the φ is continuous.

<u>Proof.</u> Define $\varphi:(0,+\infty)\to R$ by

$$\varphi(x)=\begin{cases}\varphi_o(x) & \text{for } x\in[1,+\infty),\\ \varphi_o(x+f(x))-\varphi_o(f(x)) & \text{for } x\in(0,1).\end{cases} \tag{10}$$

φ is well defined since in view of our assumption, $x \in (0,1)$ implies $f(x)>1$ (and hence $x+f(x)>1$). For the same reason if $x\in(0,1)$ then

$$\varphi(x)+\varphi(f(x))=\varphi_0(x+f(x))-\varphi_0(f(x))+\varphi_0(f(x))=\varphi_0(x+f(x))=\varphi(x+f(x)).$$

If $x=1$ then (8) holds for φ because of (9). Finally, if $x \in (1,+\infty)$ then $f(x) \in (0,1)$ and we have (keeping in mind that f is an involution)

$$\varphi(x)+\varphi(f(x))= \varphi_0(x)+\varphi_0(f(x)+f(f(x)))-\varphi_0(f(f(x)))$$

$$= \varphi_0(x+f(x))= \varphi(x+f(x)).$$

Thus φ extends φ_0 to a solution of (8). Uniqueness is obvious since every solution φ of (8) has to satisfy (10) where $\varphi_0=\varphi|_{[1,+\infty)}$. To prove the last sentence it is enough to check continuity of φ at 1. But this fact easily follows from (10) and (9).

Remark. Theorem 4 remains true if we assume that for some $a \in (0,+\infty)$ we have $f(a)=a$, $f((0,a)) \subset (a,+\infty)$, $f((a,+\infty)) \subset (0,a)$ and $\varphi_0:[a,+\infty)\to R$ atisfies $\varphi_0(2a)=2\,\varphi_0(a)$.

Proposition 3. Let $f:(0,+\infty)\to(0,+\infty)$ be continuous and decreasing surjection and let k be a positive constant. If $\varphi:(0,+\infty)\to R$ is a solution of

$$(11) \qquad \varphi(f(x)+kx)=\varphi(f(x))+\varphi(kx)$$

and the function $g:(0,+\infty)\to R$ given by $g(x)=\varphi(x)/x$ for $x \in (0,+\infty)$ is monotonic then there is a $c \in R$ such that $\varphi(x)=cx$ for $x \in (0,+\infty)$.

Proof. It is clear that g defined in our theorem satisfies

$$(12) \qquad g(f(x)+kx)= \frac{f(x)}{f(x)+kx}g(f(x))+ \frac{kx}{f(x)+kx}g(kx)$$

for every $x \in (0,+\infty)$. We will show that any monotonic solution of

(12) is constant. Without loss of generality we may assume that g is nondecreasing. For x sufficiently small we have $kx<f(x)<f(x)+kx$ as it follows from our assumptions that

$$\lim_{x\to 0+} f(x)=+\infty \quad . \tag{13}$$

Hence and by (12) we get

$$g(f(x)+kx)\leqslant \frac{f(x)}{f(x)+kx}g(f(x))+\frac{kx}{f(x)+kx}g(f(x)) = g(f(x))\leqslant g(f(x)+kx)$$

and thus $g(f(x)+kx)=g(f(x))$ for small x. Again, using (12), we infer that $g(f(x)+kx)=g(kx)$ for small x. Consequently, g has to be constant on every interval $[kx,kx+f(x)]$ provided x is small enough.

Hence and from (13) we immediately deduce that g is constant.

Remark. It is worthwhile to observe that a function $\varphi:(0,+\infty)\to(0,+\infty)$ fulfils (11) if and only if for every $x\in(0,+\infty)$ points $(0,0)$, $(kx,\varphi(kx))$, $(f(x),\varphi(f(x)))$ and $(kx+f(x),\varphi(kx+f(x)))$ are vertices of a parallelogram. This provides an immediate geometrical proof of the above thesis.

Let us conclude our considerations with a result characterizing the inverse (up to a multiplicative constant) with the help of Alsina's equation (1).

Theorem 5. Let $f:(0,+\infty)\to(0,+\infty)$ be a continuous and decreasing solution of (1). If functions $g:(0,+\infty)\to(0,+\infty)$ and $h:(0,+\infty)\to(0,+\infty)$ given for $x\in(0,+\infty)$ by

$$g(x)=f(2f(x))/x \quad \text{and} \quad h(x)=f(3f(x))/x$$

are monotonic then there exists an $a>0$ such that $f(x)=ax^{-1}$ for every $x\in(0,+\infty)$.

<u>Proof.</u> As it has been mentioned at the beginning of this paper we may assume without loss of generality that 1 is the only fixed point of f. We will prove that $f(x)=x^{-1}$ for every $x \in (0,+\infty)$. Using Lemma 1 we infer that $f^2 = id$, since f solves (2) of course. Thus as we have mentioned at the beginning of this section, $\varphi(x)=xg(x)$, $x \in (0,+\infty)$, defines a solution $\varphi:(0,+\infty) \to (0,+\infty)$ of (8) or (11) with k=1. In particular all assumptions of Proposition 3 are fulfilled. It follows that there is a $c \in R$ such that $g(x)=c$ for $x \in (0,+\infty)$. Taking into account Lemma 2 we obtain $c=g(1)=f(2f(1))=f(2)=1/2$, and hence, since f is an involution

$$f(x/2)=2f(x) \text{ for all } x \in (0,+\infty).$$

An easy induction shows that

$$f(2^n x)=2^{-n}f(x) \text{ for all } n \in Z \text{ and } x \in (0,+\infty). \tag{14}$$

Now put y=2x into (1). We get for all $x \in (0,+\infty)$

$$f(3x)+f(f(x)+f(2x))=f(f(x+f(2x)))+f(2x+f(x))). \tag{15}$$

Using (14) for $n \in \{-1,1\}$ we can write the left hand side of the above equality in the following form

$$f(3x)+f(f(x)+f(2x))=f(3x)+f(2f(2x)+f(2x))=f(3f(f(x)))+f(3f(2x)), \tag{16}$$

while the right hand side may be written in the form

$$\begin{aligned} f(f(x+f(2x))+f(f(x)+2x)))&=f(f(x+(1/2)f(x))+f(f(x)+2x)) \\ &=f(f((1/2)(2x+f(x)))+f(f(x)+2x))=f(2f(2x+f(x)) + \\ &\quad f(f(x)+2x))=f(3f(2x+f(x))). \end{aligned} \tag{17}$$

Taking into account (15), (16), (17) we see that $\varphi:(0,+\infty)\to(0,+\infty)$ given by $\varphi(x)=xh(x)$ fulfils (11) with k=2. Since h is monotonic there is a $c \in R$ such that $h(x)=c$ (cf. Proposition 3). Thus we have $c=h(1)=f(3f(1))=f(3)$ and consequently $3c=3f(3)=3h(3)=f(3f(3))=f(3c)$. Since 1 is the unique fixed point of f we obtain $3c=1$ or $c=1/3$. This together with $f^2=id$ implies

$$f(x/3)=3f(x) \text{ for all } x \in (0,+\infty),$$

whence by an obvious induction we get

(18) $\quad f(3^m x)=3^{-m}f(x)$ for all $m \in Z$ and $x \in (0,+\infty)$.

From (14) and (18) we derive (putting $x=1$) $f(2^n 3^m)=2^{-n}3^{-m}$ for all $n, m \in Z$. The set $\{2^n 3^m : n,m \in Z\}$ is dense in $(0,+\infty)$. This implies in virtue of continuity of f that $f(x)=x^{-1}$ for all $x \in (0,+\infty)$ and finishes the proof.

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