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A REMARK ON METRICS FOR FINITELY ADDITIVE DISTRIBUTION FUNCTIONS

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It is well-known that the space of distribution functions (d.f.'s) for countably additive probability measures can be metrized in such a way that the topology of the metric coincides with that of weak convergence. This can be done both when the space of d.f.'s is assumed to be Δ°, which corresponds to considering only random variables that take almost certainly real values (i.e. $P(|X|=\infty)=0$) and when r.v.'s are allowed to take values in the extended reals $\bar{R}$ (i.e. $P(|X|=\infty)\geqslant 0$); in this latter case the space Δ of d.f.'s contains Δ° properly. For Δ° the right metric is that of Lévy ([4],[5]), while for Δ one can choose in a large class of metrics (see [10],[6],[7],[9],[11],[12]).

There are, however, problems that are not amenable to a probabilistic treatment unless the probability measure used is finitely, rather than countably, additive; suffice it to mention the problem of the first digit that has recently received a great deal of attention ([2],[3],[8]).

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It is thus natural to ask whether a similar metric can be introduced on the space Δ_{fa} of finitely additive d.f.'s. We recall that a finitely additive d.f. has all the properties of a d.f. in Δ with the possible exception of right- (or left-) continuity on the reals; specifically

(1) A function $F: \bar{R} \to [0,1]$ is said to be a finitely additive distribution function if it is increasing (i.e. $x' < x''$ implies $F(x') \leqslant F(x'')$) and $F(-\infty)=0$, $F(+\infty)=1$. The space of such d.f.'s is denoted by Δ_{fa}.

This notion of d.f. is important also for the subjective foundation of probability. In fact a probability on the set of all the half-lines $[-\infty,x]$ is coherent iff it can be obtained by a finitely additive d.f. in the usual way, i.e. $P([-\infty,x])=F(x)$.

The answer to the question asked above is negative, as is easy to see; in fact it is enough to consider that the existence of a distance that metrizes weak convergence on Δ_{fa} is not compatible with the fact that weak limits of finitely additive d.f.'s are not unique. For instance the constant sequence $\{F_n\}$ with $F_n := 1_{[0,+\infty]}$ converges weakly to anyone of the d.f.'s of family $\{F_c\}$ where $F_c := 1_{]0,+\infty]} + c\, 1_{\{0\}}$; the family $\{F_c\}$ has the cardinality of the continuum. In this connexion, de Finetti's remark ought to be borne in mind that one should be free to assign to a d.f. at any of its points of discontinuity any value between its left- and its right-limit ([1]).

This negative answer is tempered by the following proposition

(2) For a set Δ^1 of Δ_{fa} the following statements are equivalent:

(a) there exists a distance d on Δ^1 that generates the topology of weak convergence;

(b) two d.f.'s of Δ^1 coincide iff they are equal at their points of continuity (i.e. for Φ_1 and Φ_2 in Δ^1, $\forall x \in C(\Phi_1) \cap C(\Phi_2)$ $(\Phi_1(x)=\Phi_2(x)) \Leftrightarrow \Phi_1=\Phi_2)$.

Proof. (a) $\Rightarrow$ (b) If two d.f.'s coincide, they obviously take the same values on the set of their points of continuity; thus it suffices to show that (a) implies the following:

if two d.f.'s in Δ^1 take the same value at their points of continuity, then they coincide (i.e. $\forall x \in C(\Phi_1) \cap C(\Phi_2)$ $(\Phi_1(x)=\Phi_2(x)) \Rightarrow \Phi_1=\Phi_2)$.

But this implication follows from the requirement that the limit be unique.

(b) $\Rightarrow$ (a) Going through the proof of [9] one sees that, for instance, the metric d_F can be defined on Δ^1 as a metric and not simply as a pseudometric; its topology is that of weak convergence.

It is a simple consequence of the above result that in order to define a metric on the space of finitely additive d.f.'s (or on one of its subsets) it is necessary to introduce a rule for assigning the value of a d.f. at every point of discontinuity: this rule is specified for countably additive d.f.'s while it can be chosen arbitrarily for finitely additive d.f.'s.

Let Δ_{max} be a maximal subset $\Delta^1 \subset \Delta_{fa}$ on which a distance d for weak convergence can be defined. Notice that Δ_{max} is composed by all the d.f.'s in Δ_{fa} with a rule for assigning a value to each d.f. at every one of its points of discontinuity; Δ_{max} does not contain Δ, the space of d.f.'s of countably additive probability measures, unless the rule assigns the right limit at each point of discontinuity. Since Helly's first theorem holds for d.f.'s in Δ_{max} in exactly the same manner as for Δ one can still conclude that the metric space (Δ_{max}, d) is compact and hence complete.

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