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INSTABILITY IN SEMI-INFINITE STRIPS OF SOLUTIONS OF LINEAR SYSTEMS

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1. Introduction.

As it is well known, the use of the energy conservation law is essential in order to apply the logarithmic convexity or weighted energy method, among others. Using them [1 to 4] it have been obtained a lot of information about the qualitative behaviour of the solutions of the equations of the ill-posed linear elastodynamic problem. But the same attention have not been observed to other conservation laws.

Recently, Knops and Stuart [5] have used a conservation law obtained by Green [6] in order to prove the uniqueness of solutions in non-linear elastostatic. A generalization of this conservation law was found by Olver [7] and it was used by the author [8] to prove instability of the classical solutions of some autonomous and homogeneous linear and non-linear equations. The present paper is devoted to study the behaviour at infinity of some solutions of non autonomous and non homogeneous second order linear systems in semi-infinite strips. Because we have explicit dependence with respect to the independent parameters of the evolution equations we lose the symmetry of the equations and thus we are not going to have conservation laws in a strict sense. Nevertheless this fact, we can obtain two evolution equalities from which, using similar methods to [8], we will obtain inestability results of the solutions at infinity.

The coordinates of a point X will be noted by $(x^1,\dots,x^n)$ and $u^i_k=(\partial/\partial x^k)u^i$. We express the total derivative respect x^j by D_j and the partial one by $(\partial/\partial x^j)$. The Einstein's summation convention will be adopted, also.

We consider a prismatic cylinder $\Omega=[0,t]\times B$ where B is a regular domain of R^{n-1} such that the boundary ∂B is sufficiently smooth to allow the divergence theorem to be applied. We denote $B(\bar{x}_1)$ the intersection of Ω with the hyperplane $x_1=\bar{x}_1$. This notation also serves to abbreviate the integral of a function over B in the following way:

$$\int_{B(\bar{x}_1)} h\,dx = \int_B h(\bar{x}_1,x_2,\dots,x_n)\,dx_2,\dots,dx_n$$

In the section two we set down two evolutionary equalities for the systems defined by the Euler-Lagrange equations of a function $W(x^i,u^j)$ in the cylinder Ω. In the section three we apply these results to obtain inestability of solutions in a semi-infinite strip.

From now on, we will suppose that the solutions are classical ones i.e. C^2-function that satisfie the differential system.

2. Evolutionary equalities. Consequences.

Let us consider the partial differential equations system given by

$$D_j(\partial W/\partial u^i_j)=0 \qquad i=1,\dots,m;\ j=1,\dots,n \tag{1}$$

in the cylinder Ω, where $W=W(x^i,u^k_j)$ is a C^2-function of their arguments. The associated boundary condition if given by

$$u^i = 0 \qquad x_i \in [0,t] \times \partial B \qquad (2)$$

We can introduce the following functions

$$P^i_j = u^k_i(\partial W/\partial u^k_j) - \delta_{ij}W$$

$$Y_j = x^i P^i_j \qquad i,j=1,\dots,n \qquad (3)$$

we have the following result

Theorem 1. Let $u(x^1,\dots,x^n)$ be a classical solution of the system (1) with the boundary conditions (2). Then we have the equalities

$$\int_{B(t)} P^1_1 dx = \int_{B(0)} P^1_1 dx - \int_0^t \left(\int_{B(\zeta)} (\partial W/\partial x^1) dx \right) d\zeta \qquad (4)$$

and

$$t \int_{B(t)} P^1_1 dx + \int_{B(t)} x^j P^j_1 dx - \int_{B(0)} x^j P^j_1 dx + \int_0^t \left(\int_{\partial B(\zeta)} N_j x^i P^i_j \, ds \right) d\zeta =$$

$$= \int_0^t \left(\int_{B(\zeta)} (u^k_i(\partial W/\partial u^k_i) - (nW + x^i(\partial W/\partial x^i))) \, dx \right) d\zeta$$

$$+ \int_0^t \left(\int_{B(\zeta)} (\partial W/\partial x^1) dx \right) d\zeta \qquad (5)$$

where the sumation in the left-hand term of the equality (5) is taken for $i,j \geqslant 2$, and N_j is the j-component of the normal vector to ∂B.

Proof: Using the equation (1), we see

$$D_j P^i_j = -(\partial W/\partial x^i)$$

and

$$D_jY_j = u_i^k(\partial W/u_i^k) - (nW+x^i(\partial W/\partial x^i))$$

Thus, employing the divergence theorem and the boundary conditions we deduce the equality (4) and

$$\int_{B(t)} x^i P_j^i dx - \int_{B(0)} x^i P_1^i\, dx + \int_0^t (\int_{\partial B(\zeta)} N_j x^i P_j^i dS) d\zeta =$$

$$= \int_0^t (\int_{B(\zeta)} (u_i^k(\partial W/\partial u_i^k) - (nW+x_i(\partial W/\partial x^i))dx)\, d\zeta$$

Introducing the equality (4) in this last equality and using that $P_j^1=0$ if $j\neq 1$ in the boundary we obtain the equality (5).

Remark. In the case that $(\partial/\partial x^1)W \leqslant 0$ the equality (4) leads to the inequality of the energy

$$\int_{B(\zeta)} P_1^1 dx \geqslant E(0)$$

and in the particular case that $(\partial/\partial x^1)W=0$ we obtain the well known energy conservation law

$$\int_{B(\zeta)} P_1^1 dx = E(0)$$

Now, we suppose that our cylinder is semi-infinite. Then $\Omega=[0,\infty)\times B$. The theorem 1 leads us to the next theorem.

Theorem 2. Let W be a function such that

(i) $u_i^k(\partial W/\partial u_i^k) - (nW+x^j(\partial W/\partial x^j)) \leqslant 0$ where $j \geqslant 2$

(ii) $(\partial/\partial x^1)W \leqslant 0$

and let B be a regular region such that

(iii) $N_j x^i P_j^i \geqslant 0$ for all $x \in \partial B$ and $i,j \geqslant 2$.

Then the classical solutions satisfy that $-\int_{B(t)} x^j P_1^j \, dx$ (where $j \geqslant 2$) goes to infinite when t goes to infinite if we suppose that $\int_{B(0)} P_1^1 dx > 0$.

<u>Proof:</u> First, let us consider the function

$$\Phi(t) = t \int_0^t \left(\int_{B(\zeta)} (\partial W/\partial x^1) dx\right) d\zeta - \int_0^t \left(\int_{B(\zeta)} \zeta \, (\partial W/\partial x^1) dx\right) d\zeta$$

This function is non-positive because $\Phi(0)=0$ and

$$\Phi'(t) = \int_0^t \left(\int_{B(\zeta)} (\partial W/\partial x^1) dx\right) d\zeta \leqslant 0.$$

Then, if we recall the equality (5) we see

$$t \int_{B(0)} P_1^1 dx + \int_{B(t)} x^j P_1^j \, dx - \int_{B(0)} x^j P_1^j dx \leqslant 0$$

where $j \geqslant 2$.

3. Dimension one.

Let us consider the case in which B is a closed interval of the real line. We can suppose that one end of the interval is zero and the other one is a real number $R>0$. We consider the linear case for the evolution system. To see the system as an evolution one, we are going to use t for the first coordinate and x for the second one. We have

$$W = a_{ij}^{lm}(t,x) u_i^l u_j^m \tag{6}$$

where i,j may be t or x and l,m lie between one to the number of unknown variables. In order that W will be a C^2 function we will suppose that the coefficients a_{ij}^{lm} are C^2-functions of their arguments.

To obtain a theorem for the behaviour at infinity we assume:

(a) There exists a positive constant δ such that

$$\int_0^R a_{xx}^{ij} u_x^i u_x^j \, dx \geq \delta \int_0^R u_x^i u_x^i \, dx$$

for all $t \geq 0$.

(b) The partial derivatives of the coefficients satisfy

$$(\partial a_{ij}^{lm}/\partial t) \, u_i^l u_j^m \leq 0 \quad \text{for all } t \geq 0$$

$$x(\partial a_{ij}^{lm}/\partial x) u_i^l u_j^m \leq 0 \quad \text{for all } t \geq 0$$

(c) There exists a positive constant k such that

$$|a_{lm}^{ij}| \leq k \qquad \text{for all } t \geq 0.$$

<u>Theorem 3.</u> We assume that W satisfies (a), (b) and (c). Let $u^i(t,x)$ be a classical solution of the problem given by (1), (2) being W given by (6). If we suppose that

$$\int_0^R a_{tt}^{ij}(0,x) u_t^i u_t^j dx > \int_0^R a_{xx}^{ij}(0,x) u_x^i u_x^j dx \tag{7}$$

Then $\int_0^R u_t^i u_t^i dx$ goes to infinite when t goes to infinite.

<u>Proof.</u> Using the Euler's formula for homogeneous function and the hypothesis (b) we have the conditions (i) and (ii) of the theorem 2.

In our case the left hand term in the condition (iii) is

$$a_{xx}^{kl} u_x^k u_x^l$$

Thus, condition (iii) is satisfied because hypothesis(a). Because

(7) is equivalent to $\int_0^R P_1^1(0,x)dx>0$ we obtain by theorem 2 that

$$-\int_0^R x\, P_1^2\, dx \to \infty \qquad \text{when} \quad t \to \infty$$

where $P_1^2 = u_x^k(a^{lk}u_t^l)$.

Now, using the arithmetic inequality, the monotonicity of the integral, the inequality of the energy and the hypothesis (a) we obtain that there exist two constants M and α such that

$$-\int_0^R xP_1^2\, dx \leqslant M \int_0^R u_t^i\, u_t^i\, dx + \alpha$$

now, the theorem is proved.

The autonomous case appear when we suppose that

$$(\partial a_{ij}^{lm}/\partial t) = 0 \tag{8}$$

In a similar way to the non autonomous case we make the following assumptions:

(a') There exists a positive constant δ such that

$$\int_0^R a_{tt}^{ij}u_t^iu_t^j \geqslant \delta\int_0^R u_t^iu_t^idx$$

(b') $a_{xx}^{ij}u_x^iu_x^j \geqslant 0$

$x(\partial a_{ij}^{lm}/\partial x)u_i^lu_j^m \leqslant 0$

We have the theorem,

<u>Theorem 4.</u> We assume that W satisfies (8), (a') and (b'). Let $u^i(t,x)$ be a classical solution of the problem given by (1), (2) being W given by (6) such that E(0)>0, then $\int_0^R u_x^iu_x^idx$ goes to infinite when t goes to infinite.

In a similar way as in theorem 3 Euler's formulae gives us the condition (i) and the condition (iii) is given by (b'). The use of arithmetic inequality, the monotonicity of the integral and energy equation proves the theorem.

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