STOCHASTICA Vol. X, Nº 2, 1986

DECOMPOSITION OF OPERATORS WITH COUNTABLE SPECTRUM(*)

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ABSTRACT

Sufficient spectral conditions for the existence of a spectral decomposition of an operator T defined on a Banach space X, with countable spectrum are given. We apply the results to obtain the West decomposition of certain Riesz operators.

1. Introduction.

If T is an operator on a Banach space X and T has a thin spectrum then certain growth conditions on the resolvent function $R_z(T)=(z-T)^{-1}$ are sufficient to ensure invariant subspaces for T, [1] , [6] , [9] and [14] . In order to obtain a spectral theory of T, a reasonable and easy to verify condition is

$$\|R_z(T)\| \leqslant (d(z,\sigma(T)))^{-1},\ z \notin \sigma(T) \qquad (1.1)$$

where $d(z,\sigma(T))$ denotes the distance of z to the spectrum $\sigma(T)$ of T, [10] . An operator which satisfies the property (1.1) is said to be a (G_1)-operator. If we extend condition (1.1) by

$$\|R_z(T)\| \leqslant K(d(z,\sigma(T)))^{-1},\ z \notin \sigma(T) \qquad (1.2)$$

(*) This paper has been partially supported by a grant from the Conselleria de Cultura,Educació i Ciència,de la Generàlitat Valenciana.

then T is said to be a (G_K)-operator. It is well known that on a Hilbert space, every hyponormal operator is a (G_1)-operator, moreover the following results are known:

(a) If T is a (G_1)-operator on a Hilbert space X, then T is isoloid, that is, every isolated point of $\sigma(T)$ is an eigenvalue of T, [11] .

(b) If X is a Hilbert space, every operator T similar to a normal operator, is a (G_K)-operator, [10] .

(c) If T is a (G_1)-operator on a Hilbert space X and $\sigma(T)$ is finite, in particular, if X is finite-dimensional, then T must be normal, [7] .

(d) In [5] , it is proved that if $\sigma(T)$ is countable and has the property that for any $z \notin \sigma(T)$ there exists some $w \notin \sigma(T)$ for which $|z-w|=d(w,\sigma(T))$, then, in general T need not be normal.

(e) Any (G_1)-operator having a countable spectrum has a normal part, [5] .

We denote L(X) the algebra of all bounded linear operators on the complex Banach space X. For T in L(X) we denote $F(T)$ the familly of all functions f which are analytic on some neighbourhood of $\sigma(T)$, where the neighbourhood can depend on $f\varepsilon F(T)$. We recall that an isolated point λ of $\sigma(T)$ is called a pole of T, if the resolvent function $R_z(T)$ has a pole at λ. By the order of a pole is meant the order of λ as a pole of $R_z(T)$ (see [3] , chapter VII). If λ is an isolated point of $\sigma(T)$ we denote $E(\lambda)$ the spectral projection associated with the point λ.

If T is an operator defined on a finite-dimensional space X with poles of order one and spectrum $\sigma(T)=\{\lambda_1,\ldots,\lambda_n\}$, then by [3], p. 561, for all f in $F(T)$ one obtain

$$f(T)=\sum_{i=1}^{n} f(\lambda_i)E(\lambda_i)$$

We recall that T in L(X) is a Riesz operator if satisfies the following properties:

(i) The non zero points of $\sigma(T)$ consist of eigenvalues of T with zero as the only possible point of accumulation.
For every complex non zero λ:

(ii) $\lambda I-T$ has finite ascent and finite descent.

(iii) For every k=1,2,..., the operator $(\lambda I-T)^k$ has closed range with finite codimension, and its kernel is finite-dimensional (see [2], chapter 3 for details).

If C is a compact operator and Q is quasinilpotent, $\sigma(Q)=\{0\}$, then clearly C+Q is a Riesz operator. Given an operator T, it is said that T admits a West decomposition if there exist operators C and Q, being C compact and Q quasinilpotent, such that T=C+Q. If X is a Hilbert space, then every Riesz operator admits a West decomposition, (see [12] ; [2] p. 51). The general problem is still unsettled for Riesz operators on Banach spaces, [2], p.50.

In [4], it is proved that a sufficient condition for the existence of a West decomposition of a Riesz operator defined on a Banach space X, is the convergence of the series $\sum_{n\geq 1} n|\lambda_n|$, where $\{\lambda_n;n=1,2,...\}$, is the set of its eigenvalues. In this paper we give different sufficient conditions on T for the existence of a West decomposition, being T a Riesz operator with poles of order one, defined on a Banach space X. Moreover, if V is an operator with countable spectrum of the type $\sigma(V)=\{0\}\cup\{\lambda_n, n=1,2,...\}$, with $\lim_{n\to\infty}\lambda_n=0$, and f belongs to $\sigma(V)$, sufficient conditions for the existence of a decomposition of the form

$$f(V)=Q+\sum_{n\geq 1} f(\lambda_n)E(\lambda_n)$$

is given.

2. Decomposition of operators with countable spectrum.

We begin showing that for operators T defined on a Banach space X, with spectrum $\sigma(T)=\{0\}\cup\{\lambda_n;\ n=1,2,\dots\}$, the (G_K)-condition (1.2) ensures a certain boundness condition related with the spectral projections of T.

<u>Lemma 1.</u> Let T be an operator on a Banach space X, with spectrum $\sigma(T)=\{0\}\cup\{\lambda_i;\ i\geq 1\}$. Suppose that T satisfies the condition (1.2). If we denote σ_n the set $\{\lambda_i;\ 1\leq i\leq n\}$, then the spectral projection $E(\sigma_n)$ satisfies $\|E(\sigma_n)\|\leq Kn$, for every $n\geq 1$.

<u>Proof.</u> Let n be a positive integer, and let $\{V_i\}_{i=1}^{n}$ be a family of neighbourhoods such that $\lambda_i \in V_i$ and $V_i\cap V_j=\emptyset$, $(\sigma(T)\sim\sigma_n)\cap V_i=\emptyset$ for $1\leq i,j\leq n$, $i\neq j$. Let V be a neighbourhood of $\sigma(T)\sim\sigma_n$, with $V\cap V_i=\emptyset$, $1\leq i\leq n$, and let γ_i be $\gamma_i(t)=\lambda_i+r_i\exp(it)$, where $t\in[0,2\pi]$, and r_i is chosen such that the image γ_i^* of γ_i, satisfies $\gamma_i^*\subset V_i$ for $1\leq i\leq n$.

Let Ω the open set $(\bigsqcup_{i=1}^{n} V_i)\cup V$, and let h_i the characteristic function of the set V_i (defined on Ω), and let $h=\sum_{i=1}^{n} h_i$, defined on Ω. By application of the Riesz-Dunford functional calculus, [3], p. 576, one obtain

$$E(\sigma_n)=\sum_{j=1}^{n} E(\lambda_j)=(1/2\pi i)\sum_{j=1}^{n}\int_{\gamma_j} h_j(z)R_z(T)\,dz \qquad (2.1)$$

From (2.1) and the (G_K)-hypothesis(1.2), it follows that

$$\|E(\sigma_n)\| = \|(1/2\pi i)\int_{\oplus_{j=1}^{n}\gamma_j} h(z)R_z(T)\,dz\| \leq (1/2\pi)\int_{\oplus_{j=1}^{n}\gamma_j}\|R_z(T)\|\,d|z|\leq$$

$$\leq n\sup_{1\leq j\leq n}\left(\sup_{z\in\gamma_j^*} d(z,\lambda_j)\|R_z(T)\|\right)\leq K\,n$$

Hence, the result is concluded.

Let T be an operator on X, and let λ be a pole of order one of T and $f\varepsilon F(T)$, then considering the Laurent expansion of the resolvent function $R_z(T)$ in the neighbourhood $0<|z-\lambda|<\varepsilon$, one obtain

$$R_z(T)=\sum_{n=-1}^{\infty} A_n(\lambda-z)^n,$$

with

$$A_{-1}=-E(\lambda)$$

(see [3], p. 573). Therefore, if we consider the Taylor expansion of f in $0\leqslant|z-\lambda|<\varepsilon$, it follows that

$$f(z)R_z(T)=E(\lambda)f(\lambda)(z-\lambda)^{-1}+\sum_{n\geqslant 0} C_n(z-\lambda)^n,$$

for certain operators C_n and for z in $0<|z-\lambda|<\varepsilon$. If we consider the circuit $\gamma(t)=\lambda+r\exp(it)$, $t\varepsilon[0,2\pi]$, and $r<\varepsilon$, then it follows that

$$\int_\gamma f(z)R_z(T)dz = f(\lambda)E(\lambda),$$

because the operator series $\sum_{n\geqslant o} C_n(z-\lambda)^n$, is analytic in a neighbourhood which contains the image of the circuit γ^*.

<u>Theorem 1.</u> Let T be an operator on a Banach space X. Suppose that its spectrum $\sigma(T)$ is of the form $\sigma(T)=\{0\}\cup\{\lambda_i;\ i=1,2,\ldots\}$, where λ_i is a pole of order one of T, and arranged so that $|\lambda_{i+1}|\leqslant|\lambda_i|$ for every $i\geqslant 1$. If T satisfies a (G_K)-condition of the type (1.2) and f is a function of the class $F(T)$ which satisfies the properties

$$f(0)=0;\quad \lim_{n\to\infty} nf(\lambda_n)=0;\quad \sum_{n\geqslant 1}|f(\lambda_{n+1})-f(\lambda_n)|n<+\infty \qquad (2.2)$$

Then there exists an operator $Q\varepsilon L(X)$ such that

$$f(T)=Q+\sum_{n\geq 1} f(\lambda_n)E(\lambda_n) \qquad (2.3)$$

<u>Proof.</u> From lemma 14.1.3, p. 301, [13], there exists sequence of neighbourhoods $\{V_n\}_{n\geq 1}$, such that $\gamma_j(t)=\lambda_j+r_j\exp(it)$, $t\varepsilon[o,2\pi]$, satisfies $\gamma_j^*\subset V_j$; V_j is a neighbourhood of λ_j, with $V_i\cap V_j=\emptyset$, if $i\neq j$. Clearly we can choose $\{V_n\}$ such that for every $j\geq 1$, there is a neighbourhood W_j of $\sigma(T)\sim\{\lambda_i;\ 1\leq i\leq j\}$, whose boundary ϕ_j is a positively oriented Jordan curve,with diameter d_j satisfying $\lim_{j\to\infty} d_j=0$. Of course we can take r_j with $\lim_{j\to\infty} r_j=0$.

Let Γ_n be the cycle $\Gamma_n=\phi_n v \bigoplus_{j=1}^{n}\gamma_j$, then by application of the Riesz-Dunford functional calculus, [3], p. 576, it follows that

$$f(T)=(1/2\pi i)\left(\int_{\phi_n} f(z)R_z(T)dz+\int_{\bigoplus_{j=1}^{n}\gamma_j} f(z)R_z(T)dz\right)$$

$$=(1/2\pi i)\left(\int_{\phi_n} f(z)R_z(T)dz+\sum_{j=1}^{n}\int_{\gamma_j} f(z)R_z(T)dz\right)$$

From the previous observation to the theorem, the last expression takes the form

$$f(T)=(1/2\pi i)\left(\int_{\phi_n} f(z)R_z(T)dz\right)+\sum_{j=1}^{n} f(\lambda_j)E(\lambda_j) \qquad (2.4)$$

Now, we prove the convergence of the operator series

$$\sum_{n=1}^{\infty} f(\lambda_n)E(\lambda_n) \qquad (2.5)$$

From the Abel transformation, [8], p. 128, it follows that

$$\sum_{k=1}^{n} f(\lambda_k)E(\lambda_k)=f(\lambda_n)E(\sigma_n)-\sum_{k=1}^{n-1} E(\sigma_k)(f(\lambda_{k+1})-f(\lambda_k)) \qquad (2.6)$$

where $\sigma_k=\{\lambda_i;\ i=1,...,k\}$. By the lemma 1, and the hypothesis (2.2) one obtain that the sequence $\{f(\lambda_n)E(\sigma_n)\}$ converges to 0 in L(X), when $n\to\infty$, and the convergence of the series

$$\sum_{k=1}^{\infty} \|E(\sigma_k)\| \ |f(\lambda_{k+1})-f(\lambda_k)|,$$

Hence, the operator series (2.5) is convergent in L(X). From (2.4) there exists the limit

$$Q = \lim_{n\to\infty} \int_{\phi_n} f(z) R_z(T) dz,$$

From (2.4), taking limits when $n\to\infty$, the result is concluded.

Taking $f(z)=z$ in the last result, we obtain the following corollary about the decomposition of (G_K)-operators with poles of order one which satisfy a spectral condition more general than $\sum_{n\leqslant 1} |\lambda_n| n<+\infty$, given in [4].

Corollary 1. Let T be an operator in L(X), X Banach, and let $\sigma(T)=\{0\} \cup \{\lambda_i;\ i\geqslant 1\}$, where λ_i is a pole of order one of T, and arranged so that $|\lambda_{i+1}| \leqslant |\lambda_i|$ for every i=1, and

$$\sum_{n\geqslant 1} |\lambda_{n+1}-\lambda_n| n <+\infty \tag{2.7}$$

If T satisfies the condition (1.2), then there exists an operator Q in L(X) such that

$$T = Q + \sum_{n\geqslant 1} \lambda_n E(\lambda_n) \tag{2.8}$$

Proof. The result is a consequence of the theorem 1, taking $f(z)=z$.

Theorem 2. Let T an operator defined on a Banach space X, which satisfies a (G_K)-condition of the type (1.2). Suppose that the spectrum $\sigma(T)$ has the form of theorem 1, and f is a fucntion of the class $\mathcal{F}(T)$, which satisfies the properties (2.2). If T is a Riesz operator, then $f(T)$ admits a West decomposition of the form

$$f(T)=Q+\sum_{n=1}^{\infty} f(\lambda_n)E(\lambda_n)$$

where Q is quasinilpotent and $C=\sum_{n\geqslant 1} f(\lambda_n)E(\lambda_n)$ is compact.

Proof. From the hypothesis of theorem 1, it follows that the operator series $\sum_{n\geqslant 1} f(\lambda_n)E(\lambda_n)$, is convergent in $L(X)$. As T is a Riesz operator, and from the properties of Riesz operators, (see [2], chapter 3), for every integer $n\geqslant 1$, the spectral projection $E(\lambda_n)$ is a finite-range operator and thus C is compact. Moreover the operator $Q=C-f(T)$ is quasinilpotent. In fact, if Q were not quasinilpotent, as Q is a Riesz operator, there would exist an eigenvalue $\lambda\neq 0$, with eigenvector $x\neq 0$. Let Y be the closed space generated by CX, we prove that $x\notin Y$. If we suppose that $x\varepsilon Y$, as $QY\subset Y$ and $Q_1=Q_{|Y}$ is a Riesz operator on Y, and for all $n\geqslant 1$, $Q_{|E(\lambda_n)X}=Q_{1|E(\lambda_n)Y}$, it follows that $Q_{1|E(\lambda_n)Y}$ is nilpotent and $(Q_1-\lambda)_{|E(\lambda_n)Y}$ is invertible.

Let p be the ascent of $Q_1-\lambda$, then $Y=R((Q_1-\lambda)^p)\oplus N((Q_1-\lambda)^p)$, where $R((Q_1-\lambda)^p)$ and $N((Q_1-\lambda)^p)$, denote the range and the nullspace of $(Q_1-\lambda)^p$. From the invertibility of $(Q_1-\lambda)^p{}_{|Y}$, it follows that $E(\lambda_n)Y\subset R((Q_1-\lambda)^p)$, for all $n\geqslant 1$, and in consequence $Y\subset R((Q_1-\lambda)^p)$.

Let Y_1 be the span of x and Y; it is clear that $f(T)Y_1\subset Y_1$ and

$$(f(T)-\lambda)Y_1=((Q_1-\lambda)+C)Y_1\subset Y$$

From here, $\lambda\varepsilon\sigma(f(T)_{|Y_1})$, and thus $\lambda\varepsilon\sigma(f(T))$. Let q be the ascent of the Riesz operator on Y_1, $f_1(T)=f(T)_{|Y_1}$, then

$$f_1(T)=R((f_1(T)-\lambda)^q)\oplus N((f_1(T)-\lambda)^q)$$

As $R(f_1(T)-\lambda)^q) \subset Y$, the subspace $N((f_1(T)-\lambda)^q)$, is not contained in Y, and thus $(Y_1\sim Y)\cap N((f(T)-\lambda)^q)\neq 0$, in contradiction with the fact $N((f(T)-\lambda)^q)\subset Y$.

Taking $f(z)=z$, in theorem 2, it follows the West decomposition of the Riesz operator T, when $\sigma(T)$ satisfies the properties of corollary 1.

Corollary 2. Let T be a (G_K)-Riesz operator on a Banach space X, and suppose that the spectrum $\sigma(T)$ satisfies the hypothesis of corollary 1, then T admits a West decomposition $T=Q+C$, of the type (2.8), where $C=\sum_{n=1}^{\infty}\lambda_n E(\lambda_n)$ is compact and Q is quasinilpotent.

Proof. This is a consequence of theorem 2 and corollary 1.

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Manuscript received in November 8, 1985.

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