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INTEGRAL EQUATIONS AND TIME VARYING LINEAR SYSTEMS

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ABSTRACT

In this paper we study the resolution problem of an integral equation with operator valued kernel. We prove the equivalence between this equation and certain time varying linear operator system. Sufficient conditions for solving the problem and explicit expressions of the solutions are given.

0. Introduction.

Let $L(X,Y)$ be the linear space of all bounded linear operators from the Banach space X into the Banach space Y and let $L(X)=L(X,X)$. When we endow this space with the strong operator topology we obtain a topological vector space which will by denoted by $L_s(X,Y)$, [14]. For the sake of clarity in the presentation we recall some concepts and properties concerning the evolution equation

$$(d/dt)u(t)=V(t)u(t), \quad t\,\varepsilon[0,T] \qquad (0.1)$$

where $u:[0,T]$---X, and $V(t)\varepsilon\ L(X)$. Given the equation (0.1), let us consider the triangle $\Delta=\{(t,s);\ 0\leqslant s\leqslant t\leqslant T\}$, we say that $U_V:\Delta\ \rightarrow L(X)$, is a fundamental operator generated by $V(t)$ if the following properties are verified:

(i) U_V is strongly continuous jointly in t,s.

(ii) The partial derivative $\partial U_V(t,s)/\partial t$, exists in the strong topology of $L(X)$, belongs to $L(X)$ for $0 \leqslant s \leqslant t \leqslant T$, and is strongly continuous in t for $(t,s)\varepsilon\Delta$.

(iii) $\partial U_V(t,s)/\partial t=V(t)U_V(t,s),(t,s)\varepsilon\Delta$, and $U_V(t,t)=I$, where I denotes the identity operator on X.

(iv) $U_V(t,s)U_V(s,u)=U_V(t,u)$, for $0 \leqslant u \leqslant s \leqslant t \leqslant T$

If $U_V(t,s)$ is invertible on Δ, then U_V may be extended to the rectangle $[0,T]^2$, in the following way

$$U_V(s,t)=(U_V(t,s))^{-1}, \quad 0\leqslant s \leqslant t\leqslant T$$

In this case, the condition (iv) is also verified for $0 \leqslant t\leqslant s \leqslant u \leqslant T$, and

$$\partial U_V(s,t)/\partial t =-(U_V(t,s))^{-1}V(t)U_V(t,s)(U_V(t,s))^{-1}=-U_V(s,t)V(t),$$
$$0 \leqslant s \leqslant t \leqslant T,$$

see [21], for details. For the finite-dimensional case $U_V(t,s)$ coincides with the transition state matrix generated by $V(t)$, [5]. Even for the infinite-dimensional case, when $V(t)=V$, one gets $U_V(t,s)=\exp((t-s)V)$. For the infinite-dimensional case, there are several known conditions for the existence of a fundamental operator U_V. Roughly speaking, there are two important cases to be distinguished: the hyperbolic and the parabolic. For the hyperbolic case $V(t)$ is for each t, the infinitesimal generator of a contraction semigroup, see [13] for the hyperbolic case, and [16] chap. 3, for the parabolic case.

In the following, integrability is always meant in the sense of Bochner (cf. [16,18]). A function $f:[0,T] \to X$, is uniformly Hölder continuous on this interval if

$$\|f(t)-f(s)\| \leqslant c\,|t-s|^a; \quad 0<a<1, \; 0 \leqslant t, \; s \leqslant T$$

where c and a are positive constants independent of t and s. In the following, P denotes a bounded projection on the Banach space X, $C \varepsilon L(X,Y)$, $A(t)\varepsilon L(X)$ and $B(t)\varepsilon L(Y,X)$, for $t\varepsilon[0,T]$, and we will suppose that $t \to A(t)$, is a continuous $L(X)$-valued operator function which generates an invertible fundamental operator $U_A(t,s)$; $B(\cdot)$ will be continuous on the interval $[0,T]$.

This paper is concerned with the resolution problem of an integral equation of the type

$$\Psi(t)=f(t)+\int_0^T K(t,s)\;\Psi(s)ds, \quad 0\leqslant t\leqslant T \tag{0.2}$$

where f and Ψ are (possibly infinite-dimensional) vector functions and K is an operator valued kernel given by

$$K(t,s)=\begin{cases} C\;U_A(t,s)\;(I-P)B(s) & ;\; t>s \\ C(U_A(s,t))^{-1}P\;B(s) & ;\; t<s \end{cases} \tag{0.3}$$

In section 1, the equivalence between the problem (0.2)-(0.3) and a time varying linear system is proved. We give conditions in order to obtain explicit expressions for solutions of the equation (0.2). Section 2 connects this problem with the resolution problem of certain operator differential equation of Riccati type. Conditions for solving (0.2) in terms of solutions of Riccati differential equations are given.

1. Integral equations and linear systems.

In the following we consider the integral equation (0.2) where T is a positive fixed number, f and Ψ are uniformly Hölder continuous functions when $A=A(t)$ and $B=B(t)$, for all t, and we suppose that f and Ψ are simply Bochner integrable for the time-independent case. For the time-varying case, we will assume that

$A(\cdot)$ and $B(\cdot)$ are continuous functions on the interval $[0,T]$.

Theorem 1.1. Let us consider the equation (0.2) whose kernel K is given by (0.3) and let us suppose that $\{A(t)\}$ is generator of a fundamental operator $U_A(t,s)$, such that

(i) $U_A(t,s)$ is invertible for all $(t,s)\varepsilon\Delta$,

(ii) $P\,U_A(t,s)=U_A(t,s)P$, $(t,s)\varepsilon\Delta$,

then the equation (0.2) and the linear system

$$\begin{aligned} &\dot{x}(t)=A(t)x(t)+B(t)u(t)\\ &y(t)=-Cx(t)+u(t)\\ &(I-P)x(0)=0,\ Px(T)=0\\ &0\leqslant t\leqslant T \end{aligned} \tag{1.1}$$

are equivalent in the following sense. If Ψ is a solution of (0.2), then the system (1.1) with input $u=\Psi$ has output $y=f$. Conversely, if the system (1.1) with input $u=\Psi$, has output $y=f$, then Ψ is a solution of (0.2).

Proof. Assume Ψ is a solution of (0.2). Define $x\colon [0,T] \to X$ by the expression

$$x(t)=\int_0^t U_A(t,s)(I-P)B(s)\Psi(s)\,ds-\int_t^T U_A(t,s)PB(s)\Psi(s)ds, \tag{1.2}$$

Then x is absolutely continuous, almost differentiable and from the properties of an invertible fundamental operator, it follows that

$$\dot{x}(t)=A(t)U_A(t,0)\int_0^t U_A(0,s)(I-P)B(s)\Psi(s)ds+U_A(0,t)U_A(0,t)(I-P)B(t)\Psi(t)-$$
$$-A(t)U_A(t,0)\int_t^T U_A(0,s)PB(s)\Psi(s)ds+U_A(t,0)U_A(0,t)PB(t)\Psi(t)=A(t)x(t)+B(t)\Psi(t)$$

As Ψ is a solution of (0.2), it follows that

$$f(t) = \Psi(t) - C\,x\,(t) \tag{1.3}$$

and as $P^2 = P$ one gets that

$$(I-P)x(0) = -(I-P)\int_o^t U_A(o,s)PB(s)\Psi(s)ds = -\int_o^T U_A(o,s)(I-P)PB(s)\Psi(s)ds$$

$$Px(T) = P\int_o^T U_A(T,s)(I-P)B(s)\Psi(s)ds = \int_o^T U_A(T,s)P(I-P)B(s)\Psi(s)ds = 0$$

Thus system (1.1) with input $u=\psi$ has output $y=f$.

Conversely, if $x:[0,T] \dashrightarrow X$, is a solution of system (1.1), with input $u=\psi$ and output $y=f$, then (1.3) is verified. As $x(t)=A(t)x(t)+B(t)\Psi(t)$, ([14], p.19), it follows the existence of some $x \varepsilon X$, such that

$$x(t) = U_A(t,0)z + \int_o^t U_A(t,s)B(s)\Psi(s)ds \tag{1.4}$$

From the boundary conditions of (1.1), it follows that $z=Pz$ and

$$Px(T) = 0 = PU_A(T,0)z + \int_o^T PU_A(T,s)B(s)\Psi(s)ds$$

and from here

$$z = Pz = -(U_A(T,0))^{-1}\int_o^T U_A(T,s)PB(s)\Psi(s)ds = -\int_o^T U_A(0,s)PB(s)\Psi(s)ds$$

Substituting the last expression of z in (1.4), we obtain (1.2). From here it follows that is a solution of (0.2)

Example 1. If X and Y are finite-dimensional Banach spaces, and P is a projection on X such that $PA(t)=A(t)P$, for all t, then $U_A(t,s)$ is the transition states matrix of the system $\dot{x}(t)=A(t)x(t)$.

From the Peano-Baker's expression of $U_A(t,s)$, [5].p.22, the commutativity between $U_A(t,s)$ and $A(t)$ is satisfied. Obviously, $U_A(t,s)$ is invertible for all $(t,s)\varepsilon\Delta$. Thus the hypotheses of theorem 1.1 are satisfied.

Example 2. If X and Y are infinite-dimensional Banach spaces with $A(t)=A$ for all t in $[0,T]$, then it is clear that $U_A(t,s)=\exp((t-s)A)$. Furthermore, if we suppose that $AP=PA$, then it follows that $PU_A(t,s)=U_A(t,s)P$, for all s,t in $[0,T]$. From here, theorem 3.1 in [4], is a particular case of theorem 1.1.

Example 3. Even for the infinite-dimensional case, if A(t) satisfies the property

$$A(t)\left(\int_o^t A(s)ds\right) = \left(\int_o^t A(s)ds\right)A(t)$$

or

$$A(t_1)A(t_2) = A(t_2)A(t_1)$$

for all t,t_1,t_2, in $[0,T]$ (see [10], p.600 for details), then it is easy to show that

$$U_A(t,s)=\exp\left(\int_0^{t-s} A(u)du\right), \quad 0\leqslant s\leqslant t\leqslant T$$

is an invertible fundamental operator generated by $\{A(t)\}$. It is clear that if $PA(t)=A(t)P$, then P and $U_A(t,s)$ commute. This situation appears, for instance, when $AP=PA$, and we consider $A(t)=A+h(t)I$, h being an analytic function of real variable t in $[-T,T]$.

By interchanging the roles of input and output, is obtained the inverse system of (1.1)

$$\begin{aligned}&\dot{x}(t)=A_1(t)x(t) + B(t)y(t)\\ &u(t)=Cx(t) + y(t)\\ &(I-P)x(0)=0, Px(T)=0\\ &t\varepsilon[0,T]\end{aligned} \qquad (1.5)$$

where $A_1(t)=A(t)+B(t)C$, it is easy to show that systems (1.1) and (1.5) are equivalent in the sense that they have the same solutions. So solving the integral equation (0.2) comes down to finding the output of (1.5) with imput y=f.

We shall now employ these connections to obtain the solutions of (0.2) through system (1.1). In the following we denote $A_1(t)=A(t)+B(t)C$.

<u>Theorem 2.1.</u> Let us consider equation (0.2) with kernel given by (0.3). Under the conditions of theorem 1.1 and the following additional one

(iii) $\{A_1(t)\}$ is a generator of a fundamental operator $U_{A_1}(t,s)$.

Then the equation (0.2) is solvable if and only if, there exists $z\varepsilon X$ such that

$$(I-P)z=0$$

$$P(U_{A_1}(T,0)z + \int_o^T U_{A_1}(t,s)B(s)f(s)ds)=0 \tag{1.6}$$

In that case the general solution of equation (0.2) is given by

$$\Psi(t)=f(t) + CU_{A_1}(t,0)z + \int_o^t CU_{A_1}(t,s)B(s)f(s)ds \tag{1.7}$$

<u>Proof.</u> Assume that Ψ is a solution of the equation (0.2). From theorem 1.1, system (1.1) with input $u=\Psi$ has output y=f. From the equivalence between system (1.1) and (1.5), taking y=f, from the first equation of (1.5) one gets $\dot{x}(t)=A_1(t)x(t)+B(t)f(t)$. From here and [14], p.19, it follows that there exists some $z\varepsilon X$ such that

$$x(t)=U_{A_1}(t.o)z + \int_o^t U_{A_1}(t,s)B(s)f(s)ds \tag{1.8}$$

As x(t) satisfies the boundary conditions of (1.5), we obtain

$$(I-P)x(o)=(I-P)U_{A_1}(0,0)z=(I-P)z=0$$

$$Px(T)=0=P(U_{A_1}(T,0)z+\int_o^T U_{A_1}(T,s)B(s)f(s)ds)$$

Thus conditions (1.6) are verified. Otherwise, we have $\Psi(t)=Cx(t)+f(t)$, that is, Ψ is given by (1.7).

Conversely, assume that Ψ is given by (1.7), where z satisfies (1.6). If we define x(t) by (1.8) for $t\varepsilon[0.T]$, then x is absolutely continuous, almost everywhere differentiable (everywhere differentiable for the time varying case from the continuity of B(.) and the Hölder continuity of f(t)), and $\dot{x}(t)=A_1(t)x(t)+B(t)f(t)$. As Ψ is given by (1.7) we have $\Psi(t)=Cx(t)+f(t)$. It is clear that from the hypothesis (1.6) on vector z implies that x satisfies the boundary conditions of (1.5). Thus, from the equivalence between (1.1) and (1.5) and theorem 1.1, it follows that Ψ is a solution of (0.2).

The following corollary yields explicit expression for the general solution of (0.2) in the homogeneous case.

Corollary 1.3. Under the hypothesis of theorem 1.2, and for the case f=0, the general solution of (0.1) is given by

$$\Psi(t)=CU_{A_1}(t,0)z, \quad 0 \leqslant t \leqslant T \tag{1.9}$$

where z is a vector from X satisfying

$$(I-P)z=0, \quad PU_{A_1}(T,0)z=0 \tag{1.10}$$

For the time-invariant case, that is A(t)=A and B(t)=B for all t in [0,T] the result of theorem 1.2, and corollary 1.3, generalize theorem 3.2 and corollary 3.3 of 4, respectively. About the hypothesis (iii) of theorem 1.2, several conditions can be imposed on a generator $\{A(t)\}$ of a fundamental operator $U_A(t,s)$, in order that $A_1(t)=A(t)+B(t)C$ will be generator of a fundamental operator. (See [11], [12], [13], [14], and [16]). In particular if $\{B(t)C\}$ is uniformly bounded and $\{A(t)\}$ is generator of certain type, then $A_1(t)=A(t)+B(t)C$, is also generator of a fundamental operator, see the appendix of [6] for details. Otherwise, it is obvious that for the finite-dimensional case and for the time invariant case, no problem exists about this matter.

2. Integral equations and Riccati operator equations.

Let X_1 and X_2 be complex Banach spaces and let us consider the generalized operator Riccati differential equation

$$\left.\begin{array}{r} (d/dt)U(t)=A(t)+B(t)U(t)-U(t)C(t)-U(t)D(t)U(t) \\ U(0)=F \end{array}\right\} \qquad (2.1)$$

where $B(t)\varepsilon L(X_1,X_1)$, $A(t)\varepsilon L(X_2,X_1)$, $D(t)\varepsilon L(X_1,X_2)$, $C(t)\varepsilon L(X_2,X_2)$ and $F\varepsilon L(X_2,X_1)$, $U(t)\varepsilon L(X_2,X_1)$. Locally, in a open neigbourhood of 0, problem (2.1) is always uniquely solvable [14]. The next theorem deals with the global solution of (2.1) in $[0,T]$. In the following we denote $X=X_1\oplus X_2$, and we represent $W(t)\varepsilon L(X,X)$ by the operator

$$W(t)=\begin{bmatrix} B(t) & A(t) \\ D(t) & C(t) \end{bmatrix} \qquad (2.2)$$

<u>Theorem 2.1.</u> Let us consider problem (2.1) where $\{W(t)\}$ is generator of an invertible fundamental operator $U_W(t,s)$

$$U_W(t,s)=\begin{bmatrix} U_{11}(t,s) & U_{12}(t,s) \\ U_{21}(t,s) & U_{22}(t,s) \end{bmatrix} \qquad (2.2)$$

Then problem (2.1) is solvable in $[0,T]$, if and only if

$$S(t)=U_{21}(t,0)F+U_{22}(t,0),\ \text{is invertible on } [0,T] \quad (2.3)$$

In this case, the unique solution of (2.1) is given by

$$U(t)=[I,0]U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}=$$

$$=(U_{11}(t,0)F+U_{12}(t,0))(U_{21}(t,0)F+U_{22}(t,0))^{-1} \quad (2.4)$$

<u>Proof.</u> Suppose that $S(t)$ is invertible for all t in $[0,T]$.
As $S(t)\varepsilon L(X_2,X_2)$, $S(t)=[0,I]U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}$, by differentiation in the expression (2.4), it follows that

$$(d/dt)U(t)=[I,0]\{(d/dt)U_W(t,0)\}\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}+[I,0\begin{bmatrix}F\\I\end{bmatrix}d/dt(S(t))^{-1}=$$

$$=[I,0]\,W(t)U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}-[I,0]U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}\{(d/dt)(S(t))\}(S(t))^{-1}=$$

$$=[I,0]W(t)U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}-[I,0]U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}[0,I]W(t)U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}=$$

$$=[B(t),A(t)]U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}-U(t)[0,I]W(t)U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}(S(t))^{-1}=$$

$$=B(t)\,U(t)+A(t)-U(t)C(t)-U(t)D(t)U(t).$$

Thus, $U(t)$ is a solution of (2.1) on $[0,T]$, since $U(0)=F$.

Conversely, let us assume that $U(t)\varepsilon L(X_2,X_1)$ is a solution of (2.1) on $[0,T]$. If we define $L(t)=U(t)[0,I]\,U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}-$ $-[I,0]\,U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}$, for all t in $[0,T]$. It is clear that $(d/dt)\,L(t)=[B(t)-U(t)D(t)]L(t)$, for all t on $[0,T]$, and $L(0)=0$. From here, $L(t)=0$, $t\varepsilon[0,T]$.

Thus one gets

$$U(t)[0,I]\ U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}=[I,0]\,U_W(t,0)\begin{bmatrix}F\\I\end{bmatrix}\ ;\ t\varepsilon[0,T] \quad (2.5)$$

By using $U_W(0,t) = (U_W(t,0))^{-1}$, for t in $[0,T]$ it follows

$$[0,I]\, U_W(0,t) \begin{bmatrix} U(t) \\ I \end{bmatrix} S(t) = [0,I] U_W(0,t) \begin{vmatrix} 0 & U(t) \\ 0 & I \end{vmatrix} U_W(t,0) \begin{vmatrix} F \\ I \end{vmatrix} =$$

and from (2.5) the last expression coincides with

$$[0,I] U_W(0,t) \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix} U_W(t,0) \begin{vmatrix} F \\ I \end{vmatrix} = [0,I] \begin{vmatrix} F \\ I \end{vmatrix} = I_{X_2}.$$

Thus $S(t)$ is left invertible for all t in $[0,T]$. For $t=0$, we obtain $S(0)=I_{X_2}$ and since $t \to S(t)$ is continuous, it follows that $S(t)$ is invertible for all t in $[0,T]$, and $(S(t))^{-1}$ is given by

$$(S(t))^{-1} = [0,I] U_W(0,t) \begin{vmatrix} U(t) \\ I \end{vmatrix}$$

Example 4. If X_1 and X_2 are finite-dimensional Banach spaces, taking $U_W(t,s)$ the transition states matrix of the linear system with coefficient matrix $W(t)$, the result of theorem 2.1 coincides with theorem 1 of [17].

Example 5. If we consider the time-invariant case, $A=A(t)$, $B=B(t)$, $C=C(t)$ and $D=D(t)$, for all t in $[0,T]$ then $\exp((t-s)W)$, is an invertible fundamental operator generated by $W=\begin{vmatrix} B & A \\ D & C \end{vmatrix}$, and theorem 2.1 coincides with lemma 8.1 of [4].

The following application of theorem 2.1 is an infinite-dimensional time-varying generalization of th.2 of [9] and results of [15] concerned with the resolution problem of two point boundary value problems for time-varying Riccati operator differential equations.

Corollary 2.2. Let us consider the problem

$$\left.\begin{array}{ll} (d/dt)U(t) = A(t)+B(t)-U(t)C(t)-U(t)D(t)U(t) & \\ t\varepsilon[0,T] \qquad PU(T) - U(0)Q = G & \end{array}\right\} \qquad (2.6)$$

Where $P\varepsilon L(X_1,X_1)$, $Q\varepsilon L(X_2,X_2)$ and $G\varepsilon L(X_2,X_1)$. Under the hypothesis of th. 2.1 the problem (2.6) is solvable if and only if, the algebraic operator equation

$$R + SZ - ZT - Z\,V\,Z = 0 \tag{2.7}$$

is solvable, where

$$\begin{aligned} R &= PU_{12}(T,0) - GU_{22}(T,0), \quad S=PU_{11}(T,0)-GU_{21}(T,0) \\ T &= Q\,U_{22}(T,0) \qquad\qquad\qquad\qquad V=Q\,U_{21}(T,0) \end{aligned} \tag{2.8}$$

Proof. Assume that $U(.)$ is a solution of (2.6). From the boundary condition one gets

$$P\{U_{11}(T,0)F+U_{12}(T,0)\}\{U_{21}(T,0)F+U_{22}(T,0)\}^{-1}-FQ = G;\ U(0)=F=Z$$

and postmultiplying by $U_{21}(T,0)F+U_{22}(T,0)$, it follows

$$(PU_{12}(T,0)-GU_{22}(T,0))+(PU_{11}(T,0)-GU_{21}(T,0))Z-ZQU_{22}(T,0)-ZQU_{21}(T,0)Z=0$$

that is, equation (2.7) is solvable and $Z=U(0)$ is a solution.

Conversely, if Z is a solution of (2.7), taking $U(0)=F=Z$ and by application of theorem 2.1 we obtain $U(\cdot)$, which satisfies (2.1) and (2.6).

About the resolution problem of the algebraic Riccati operator equation (2.7) there are several known results, see for instance, [7],[2], p.118; [8].

The following theorem is related to theorem 1 of [1] and theorem 4.1 of [4] . Its is interested in finding sufficient conditions for the existence of a solution (unique) of equation (0.2), as well as to find explicit expressions of the solution and the resolvent kernel of (0.2).

<u>Theorem 2.3.</u> Consider equation (0.2) where K is given by (0.3) in such a way that $\{A(t)\}$ and $\{A_1(t)\}$, with $A_1(t)=A(t)+B(t)C$, are generators of invertible fundamental operators $U_A(t,s)$ and $U_{A_1}(t,s)$, such that $U_A(t,s)$ conmutes with P, for all s, t in $[0,T], s<t$. Let S_T be the operator

$$S_T = PU_A(T,0)U_{A_1}(T,0)\Big|_{\operatorname{Im} P} : \operatorname{Im} P \to \operatorname{Im} P \tag{2.9}$$

If S_T is invertible then (0.2) has only one Hölder continuous solution for each Hölder continuous function f. In this case, the unique solution of the equation (0.2) can be written as

$$\Psi(t)=f(t) + \int_o^T R_T(t,s)f(s)ds, \quad 0\leqslant t\leqslant T \tag{2.10}$$

where

$$R_T(t,s)=\begin{cases} CU_{A_1}(t,0)\Pi_T U_{A_1}(0,s)B(s), & s<t \\ -CU_{A_1}(t,0)(I-\Pi_T)U_{A_1}(0,s)B(s), & s>t \end{cases} \tag{2.11}$$

and Π_T is the projection of X along Im P, defined by

$$\Pi_T z = z - S_T^{-1}P\, U_A(T,0)U_{A_1}(T,0), \quad z\varepsilon X \tag{2.12}$$

<u>Proof.</u> From the conmutativity between P and $U_A(t,s)$, and the invertibility of this fundamental operator, it is a straightforward matter to show that condition (1.6) of th.2.1 can be written as

$$z\varepsilon \operatorname{Im}P, \quad S_T z=-PU_A(T,0)\int_o^T U_{A_1}(T,s)B(s)f(s)ds \tag{2.13}$$

and the condition (1.10) of corollary 1.3, is equivalent to the condition $z\varepsilon \operatorname{Ker} S_T$. If S_T is injective, then the condition (1.10) implies that z=0. From corollary 1.3 the homogeneous equation associated with (0.2) has only the trivial solution. Hence, if (0.2) is solvable, this equation has only one Holder continuous solution. Furthermore, if S_T is a surjective and f is Hölder continuous, it is clear the existence of some $z \in \mathrm{Im}P$, satisfying the condition (2.13). From th. 2.1 the fucntion Ψ given by (1.7) is a solution of (0.2). Thus the first part of the theorem is proved.

From the definition, it is clear that Π_T satisfies $\Pi_T z=0$ for all $z \in \mathrm{Im}P$, and $\Pi_T(\Pi_T z)=\Pi_T z - \Pi_T(S_T^{-1}S_T z)=\Pi_T z$, for all $z \in X$. So Π_T is a projection. Moreover $\Pi_T z=0$ implies $z \in \mathrm{Im}\, P$, hence Π_T is a projection along Im P.

If f is a Hölder continuous function on $[0,T]$, and we take

$$z=-(I-\Pi_T)\int_0^T U_{A_1}(0,s)B(s)f(s)ds, \tag{2.14}$$

then (2.13) is satisfied. From th. 2.1, the function defined on $[0,T]$, by the expression

$$\begin{aligned}\Psi(t)&=f(t)-CU_{A_1}(t,0)(I-\Pi_T)\int_0^T U_{A_1}(0,s)B(s)f(s)ds\ +\\ &\quad+\int_0^t C\ U_{A_1}(t,s)B(s)f(s)ds\ =\\ &=f(t)-C\ U_{A_1}(t,0)\int_0^t (I-\Pi_T)U_{A_1}(0,s)-U_{A_1}(0,s)B(s)f(s)ds\ -\\ &\quad-\int_t^T C\ U_{A_1}(t,0)(I-\Pi_T)U_{A_1}(0,s)B(s)f(s)ds\end{aligned}$$

and thus (2.10) is verified.

The following proposition provides a characterization of the invertibility of the operator S_T given by (2.9), in terms of the global solvability of certain Riccati operator differential equation. Let us consider the equation (0.2) with kernel K given by (0.2). Put $X_1=\mathrm{Ker}\ P$ and $X_2=\mathrm{Im}\ P$. With respect to the decomposition $X=X_1 \oplus X_2$, we write the operator $A_1(t)=A(t)+B(t)C$ as

$$A_1(t)=\begin{vmatrix} A_{11}(t) & A_{12}(t) \\ A_{13}(t) & A_{22}(t) \end{vmatrix} : X_1 \oplus X_2 \to X_1 \oplus X_2 \tag{2.15}$$

<u>Proposition 2.4.</u> Let us assume that $\{A_1(t)\}$ is generator of an invertible fundamental operator $U_{A_1}(t,s)$, $0 \leq s \leq t \leq T$, and $\{A(t)\}$ is generator of a fundamental operator $U_A(t,s)$, $0 \leq s \leq t \leq T$. Then the operator S_T given by (2.9) is invertible, if and only if, $U_A(T,0)$ is invertible and the Riccati operator differential equation

$$(d/dt)U(t)=A_{12}(t)+A_{11}(t)U(t)-U(t)A_{14}(t)-U(t)A_{13}(t)U(t)$$
$$t \in [0,T] \;, U(0)=0 \tag{2.16}$$

is solvable on $[0,T]$.

<u>Proof.</u> From th. 2.1, the problem (2.16) is solvable on $[0,T]$, if and only if, $H(t)=[0,I]\; U_{A_1}(t,0) \begin{bmatrix} 0 \\ I \end{bmatrix}$, is invertible on $[0,T]$. If we consider the operator function $H(\cdot)$ defined on $0,T$, by the expression

$$H(t)=[0,I]\; U_{A_1}(t,0) \begin{vmatrix} 0 \\ I \end{vmatrix} : X_2 \text{ --- } X_2$$

then it follows that

$$S_T=[0,I]\; U_A(T,0) \begin{vmatrix} 0 \\ I \end{vmatrix} H(T)$$

From the invertibility of $U_A(T,0)$ and the hypothesis, the proposition is proved.

The results obtained in this section may be regarded as a generalization of those obtained for the scalar in [1], [20], and theorem 8.4 in [4] where the time invariant operator case is studied.

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