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WEIGHTED SHIFT OPERATORS ON ℓ_p SPACES

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ABSTRACT

The analytic-spectral structure of the commutant of a weighted shift operator defined on a ℓ_p space $(1 \leq p < \infty)$ is studied. The cases unilateral, bilateral and quasinilpotent are treated. We apply the results to study certain questions related to unicellularity, strictly cyclicity and the existence of hyperinvariant subspaces.

1. Introduction.

For $1 \leq p < \infty$ let $\ell_p(Z)$ be the Banach space of all absolutely p-summable sequences of complex numbers $x=\{x_n\}_{n \in Z}$ with the norm

$$\|x\| = \|x\|_p = \{\sum_{n \in Z} |x_n|^p\}^{1/p}$$

where Z is the set of all integers. For the case $p=\infty$ let $\ell_\infty(Z)$ be the Banach space of all bounded sequences of complex numbers $x=\{x_n\}_{n \in Z}$ with the norm

$$\|x\| = \|x\|_\infty = \sup_{n \in Z} |x_n|$$

An analogous definition is given for the spaces $\ell_p(N)$, where N is the set of all positive integers and $1 \leq p \leq \infty$. In the following we will denote the spaces $\ell_p(I)$ where the set I will be the set Z or N.

If $\{w_n\}_{n \in I}$ is a bounded sequence of nonzero complex numbers, then the operator T, defined on $\ell_p(I)$ by the expression

$$Te_n = w_n e_{n+1} \quad (n \in I) \tag{1.1}$$

where $\{e_n\}_{n \in I}$ is the natural basis of $\ell_p(I)$, is called a weighted shift operator on $\ell_p(I)$. If I=N, T is said to be an unilateral weighted shift operator on $\ell_p(N)$, and T is said to be a bilateral weigthed shift on $\ell_p(Z)$, when I=Z. We may assume without loss of generality that for all $n \in I$, $w_n \geq 0$, [10]. Let $\{\beta(n)\}_{n \in I}$ a sequence of positive numbers with $\beta(0)=1$, then we define the spaces

$$L_p(\beta)=\{(f(n))_{n \in Z};\ (f(n)\beta(n))_{n \in Z} \text{ is absolutely p-summable}\},\quad 1 \leq p < \infty.$$

$$L_\infty(\beta)=\{(f(n))_{n \in Z};\ (f(n)\beta(n))_{n \in Z} \text{ is bounded}\}$$

$$H_p(\beta)=\{(f(n))_{n \in N};\ (f(n)\beta(n))_{n \in N} \text{ is absolutely p-summable}\},\quad 1 \leq p < \infty.$$

$$H_\infty(\beta)=\{(f(n))_{n \in N};\ (f(n)\beta(n))_{n \in N} \text{ is bounded}\}$$

we shall use the notation

$$f(z) = \sum_{n \in I} f(n) z^n \tag{2.1}$$

where the series can be convergent or not for any complex value of z. These spaces are Banach spaces with the norm

$$\begin{aligned} \|f\|_p &= \{\sum_{n \in I} |f(n)|^p \beta(n)^p\}^{1/p}, \quad 1 \leq p < \infty \\ \|f\|_\infty &= \sup_{n \in I} |f(n)|\, \beta(n), \quad p=\infty \end{aligned} \tag{3.1}$$

Let $f_k(n)=\delta_{nk}$; in the notation of equation (2.1) we have $f_k(z)=z^k$. Then $\{f_k\}_{k \in Z}$ is a basis of $L_p(\beta)$, and $\{f_k\}_{k \in N}$ is a basis of $H_p(\beta)$, and from (3.1) it follows that

$$\| f_k \| = \beta(k) \qquad (4.1)$$

Now we consider the linear transformation M_z of multiplication by z on the spaces $L_p(\beta)$ and $H_p(\beta)$:

$$(M_z f)(z) = \sum_{n \in I} f(n) z^{n+1} \qquad (5.1)$$

If X is a Banach space we denote by L(X) the set of all bounded linear operators defined on X with the operator norm, [5]. If T lies in L(X), its spectrum will be denoted by $\sigma(T)$. We recall that an operator T in L(X) is quasi-nilpotent if $\sigma(T)=\{0\}$. An operator T in L(X) is said to be unicellular if its lattice of invariant subspaces represents a bounded linear operator on a Banach space.

The commutant of an operator is the set of all those operators that commute with him. In section 2, by using Banach algebras and analytic function theory, the analytic spectral structure of the commutant of an invertible bilateral weighted shift operator on $\ell_p(Z)$ is studied. We apply the results to obtain spectral conditions for unicellularity of T. Known results of [15], obtained*for the case p=2 are extended, and sufficient conditions for the existence of hyperinvariant subspaces are given. We recall that a subspace M is said to be hyperinvariant for T if M is an invariant subspace for every operator which commutes with T. Section 3 is concerned with the study of the analytic-spectral structure of the commutant of an unilateral weighted shift operator on $\ell_p(N)$, $1 \leq p \leq \infty$. Spectral conditions for unicellularity are found. The case where T is a quasi-nilpotent operator is studied separately. Weighted shift operators with monotone weight sequence are analyzed in section 3, where results about strictly cyclycity of operators obtained in [9] for the case p=2 are extended.

2. On bilateral weighted shift operators on ℓ_p spaces.

An operator T is said to shift a basis $\{f_n\}_{n\in Z}$ if $Tf_n=f_{n+1}$, for all $n\in Z$. The following theorem will allow us to represent an injective bilateral shift operator as an ordinary shift operator on a Banach space of formal Laurent series.

<u>Theorem 1.</u> Let T be an operator on $\ell_p(Z)$, $1\leqslant p\leqslant\infty$, then it follows that:

(i) T is an injective bilateral weighted shift on $\ell_p(Z)$ if and only if T shifts some basis of $\ell_p(Z)$.

(ii) If T is an injective bilateral weighted shift on $\ell_p(Z)$ then T can be represented as the operator M_z acting on $L_p(\beta)$, for a suitable sequence β. The relation between $\{w_n\}_{n\in Z}$ and β is given by

$$\begin{aligned} &\beta(n)=w_o w_1\dots w_{n-1} && (n>0)\\ &\beta(0)=1 && \\ &\beta(-n)=(w_{-1}\dots w_{-n})^{-1} && (n<0)\\ &w_n=\beta(n+1)(\beta(n))^{-1} && (n\in Z)\end{aligned} \tag{1.2}$$

(iii) $T=M_z$ on $L_p(\beta)$ is bounded if and only if $\{w_n\}_{n\in Z}$ is bounded, and in this case

$$\|M_z^n\| = \sup_{k\in Z} w_k w_{k+1}\dots w_{k+n-1},\quad n>0$$

(iv) If $T=M_z$ on $L_p(\beta)$, then T is invertible if and only if $\{(\beta(j+1))^{-1}\beta(j)\}_{j\in Z}$ is bounded. In this case

$$\|M_z^{-n}\| = \sup_{k\in Z}\beta(k+n)(\beta(k))^{-1}\quad (n>0)$$

Proof. (i) If T shifts some basis $\{f_n\}_{n\in Z}$, then it follows that $Te_n=w_ne_{n+1}$, where

$$e_n=f_n(\|f_n\|)^{-1},\ w_n=\|f_{n+1}\|\ (\|f_n\|)^{-1}\quad (n\in Z)$$

Conversely, let T be an injective bilateral shift operator defined by the expression (1.1), where $\{e_n\}_{n\in Z}$ is the natural basis of $\ell_p(Z)$. Then T shifts the basis $\{f_n\}_{n\in Z}$, where $f_o=e_o$ and

$$f_n=(w_ow_1\ldots w_{n-1})e_n$$

$$f_{-n}=(w_{-n}w_{-n+1}\ldots w_{-1})e_{-n}$$

for all $n\geqslant 0$.

(ii) It is a consequence of (i). (iii) From the expression (1.1) it follows that for all $k\in Z$,

$$T^ne_k=(w_kw_{k+1}\ldots w_{k+n-1})e_{n+k}$$

From here and (1.2) the result is deduced.

(iv) Since $M_zf_m=f_{m+1}$, we have $M_z^{-1}f_m=f_{m-1}$, for all $m\in Z$, where M_z^{-1} is defined on finite linear combinations of basis vectors. Thus the linear transformation M_z^{-1} shifts the basis $\{g_n\}_{n\in Z}$, where $g_n=f_{-n}$, for all $n\in Z$. Hence we represent T^{-1} as multiplication by z on $L_p(\beta')$, where $\beta'(n)=\|g_n\|=\|f_{-n}\|=\beta(-n)$, for all $n\in Z$. From (iii) M_z^{-1} is bounded if and only if $\{\beta'(n+1)(\beta'(n))^{-1}\}_{n\in Z}$ is bounded, and the result is concluded from (iii) and the expression which defines β'.

Now we consider the multiplication of formal Laurent series $h=fg$

$$\Big(\sum_{n\in Z} f(n)z^n\Big)\Big(\sum_{n\in Z} g(n)z^n\Big)=\sum_{n\in Z} h(n)z^n \qquad (2.2)$$

where

$$h(n) = \sum_{k \in Z} f(k)g(n-k) \qquad (3.2)$$

and the product (2.2) is defined only all the series (3.2) are convergent. Let $L_p^\infty(\beta)$ the set of formal Laurent series $\phi(z) = \sum_{n \in Z} \phi(n) z^n$ such that $\phi L_p(\beta) \subset L_p(\beta)$. Note that if $\phi \in L_p^\infty(\beta)$ then ϕf is defined and lies in $L_p(\beta)$ for all f in $L_p(\beta)$. Since $1 = f_o \in L_p(\beta)$ and $\phi f_o = \phi$ for all Laurent series ϕ, we see that $L_p^\infty(\beta) \subset L_p(\beta)$. If $\phi \in L_p^\infty(\beta)$ then the linear transformation of multiplication by ϕ on $L_p(\beta)$ will be denoted by M_ϕ.

Theorem 2.

(i) If $\phi \in L_p^\infty(\beta)$ then M_ϕ is a bounded linear transformation on $L_p(\beta)$.

(ii) If $\phi, \psi \in L_p^\infty(\beta)$, then their product is defined and is an element of $L_p^\infty(\beta)$ and $M_\phi M_\psi = M_{\phi\psi}$.

(iii) If B is an operator on $L_p(\beta)$ that commutes with M_z then $B = M_\phi$, for some $\phi \in L_p^\infty(\beta)$.

(iv) With the norm $\|\phi\| = \|M_\phi\|$, the space $L_p^\infty(\beta)$ is a commutative Banach algebra with unit.

Proof. (i) Let $\{f_k\}$ be a basis of $L_p(\beta)$ with $f_k(z) = z^k$, for all $k \in Z$. If we define p_n by $p_n(g) = g(n)$, for all $n \in Z$, and $g \in L_p(\beta)$, then the n-th component of $M_\phi f_m = \phi f_m$ with respect to the basis $\{f_k\}$ satisfies

$$f_n^*(\phi f_m) = \phi(n-m)(\beta(n))^2 = p_n(\phi f_m) \qquad (4.2)$$

where $\{f_k^*\}$ denotes the dual basis of $\{f_k\}$. In fact, it follows that

$$z^m\phi(z)=\sum_{k\in Z}\phi(k)z^{k+m}=\sum_{k\in Z}\phi(k-m)z^k,\quad m\in Z \qquad (5.2)$$

If $\{e_k\}$ is the natural basis of $\ell_p(Z)$, the from theorem 1 it follows that

$$f_k=\beta(k)e_k$$

$$f_m^*(f_n)=(\beta(m)e_m)(\beta(n)e_n)=\beta(m)\beta(n)\delta_{mn}$$

$$f_m^*(f_m)=(\beta(m))^2$$

From (4.2), the matrix (a_{nk}) which represents the transformation with respect to the basis $\{f_k\}$ is given by $a_{nk}=\phi(n-k),n,k\in Z$. We note that the matrix is lower triangular, $a_{nk}=0$ for $k>n$. From [3], this matrix transformation is bounded.

(ii) By (i), M_ϕ and M_ψ are bounded operators, whose matrix with respect to the basis $\{f_k\}$ have (i,j)-th entries $\phi(i-j)$ and $\psi(i-j)$, respectively. Thus the product operator $M_\phi M_\psi$ is represented by the product of these two matrix. Otherwise, the series that arise in the product matrix are all convergent and the (i,j)-th entry in the product matrix is

$$\sum_{n\in Z}\phi(i-n)\psi(n-j)=\sum_{k\in Z}\phi(k)\psi(i-j-k)$$

which is precisely the formula for $p_{(i-j)}(\phi\psi)=(\phi\psi)(i-j)$ by (2.2). Thus it follows that $M_\phi M_\psi=M_{\phi\psi}$.

(iii) Let $\phi=Bf_o$. Then by commutativity it follows that

$$Bf_k=BM_z^kf_o=M_z^kBf_o=z^k\phi=\phi\ \ f_k \qquad (k\geqslant 0)$$

$$z^{-k}(Bf_k)=M_z^{(-k)}Bf_k=Bf_o=\phi \qquad (k<0)$$

By continuity of p_n and the convergence of $g=\sum_{k\in Z}g(k)z^k$ in $L_p(\beta)$ it follows that

$$p_n(Bg)=\sum_{k\in Z} g(k)p_n(Bf_k)=\sum_{k\in Z} g(k) \quad (n-k)$$

which converges for all $n \in Z$.

(iv) It is easy to prove that the commutant $L_p^\infty(\beta)$ is closed in the weak operator topology of $L(L_p(\beta))$. From [5], p. 477, it follows that $L_p^\infty(\beta)$ is closed in the norm of $L(L_p(\beta))$ and thus it is a Banach space.

In the following we identify the commutant of T with the space $L_p^\infty(\beta)$. The following definition is concerned with the next result.

Definition 1. Let $T=M_z$ on $L_p(\beta)$ and let $w\neq 0$ be a complex number; λ_w denotes the functional of evaluation at w, defined on Laurent polynomials by $\lambda_w(p)=p(w)$; w is said to be a bounded point evaluation on $L_p(\beta)$ if the functional λ_w extends to a bounded linear functional on $L_p(\beta)$. In this case f(w) will denote the complex number $\lambda_w(f)$, for f in $L_p(\beta)$.

Let $K(L_p^\infty(\beta))$ be the structure space of $L_p^\infty(\beta)$, then the functional $\rho:K(L_p^\infty(\beta)) \to C$; $\rho(\lambda)=\lambda(z)$, satisfies $\lambda(z)\in\sigma(T)$, [2], p. 233. Thus the only bounded points evaluation lie in $\sigma(T)$.

Theorem 3. Let $T=M_z$ be an invertible bilateral weighted shift on $L_p(\beta)$, with $1\leqslant p<\infty$, then if the symbol $r(\cdot)$ denotes the spectral radius, [4], it follows that

(i) $\sigma(T)=\{z; r(T^{-1})^{-1}\leqslant |z| \leqslant r(T)\}$.

(ii) If f lies in $L_p^\infty(\beta)$, and $S_{n,m}(f)=\sum_{-n\leqslant k\leqslant m} f(k)z^k, n,m\geqslant 0$ satisfies

$$S_{n,m}(f) \text{ converges to } f\text{, with respect the norm of } L_p^\infty(\beta) \text{ when } n,m\to+\infty, \tag{6.2}$$

then $\sigma(f)=\{f(w); w\in\sigma(T)\}$.

(iii) If f lies in $L_p^\infty(\beta)$ and condition (6.2) is satisfied, then the Laurent series (2.1) defines a continuous function in $\sigma(T)$.

(iv) For $p>1$, w is a bounded point evaluation on $L_p(\beta)$ if and only if

$$\sum_{n\in Z} |w|^{qn}(\beta(n))^{-q} < \infty \qquad (1/p + 1/p = 1) \qquad (7.2)$$

If $p=1$, w is a bounded point evaluation if and only if

$$\{|w|^n(\beta(n))^{-1}\}_{n\in Z} \text{ is bounded} \qquad (8.2)$$

(v) If $p>1$ and $|w|=\|T\|$ or $|w| = \|T^{-1}\|^{-1}$, then w is not a bounded point evaluation on $L_p(\beta)$.

(vi) If $\phi \in L_p(\beta)$ and $g \in L_p(\beta)$, then $\lambda_w(\phi g)=\lambda_w(\phi)\lambda_w(g)$.

<u>Proof.</u> (i) If c is a complex number with $|c|= 1$, then T is unitarily equivalent to the operator cT, [10], and thus the spectrum of T, $\sigma(T)$ has circular symmetry. Let w be in the resolvent set of T. From the series representation of the resolvent function of T, [5], p. 567, it is clear that $(T-w)^{-1}$ commutes with T. From theorem 2-(iii) there exists $\phi \in L_p^\infty(\beta)$ such that $(T-w)^{-1}=M_\phi$. Computing in the expression $(z-w)\phi(z)=1$, it follows that $\phi(-1)-w\phi(0)=1$, and

$$w^k\phi(k)=\phi(0), \quad \phi(-k-1)=w^k\phi(-1) \qquad (k \geqslant 0) \qquad (9.2)$$

From (8.2) it follows that the n-th component of $f_m\phi$ in the natural basis of $L_p(\beta)$ is given by

$$f_n^*(f_m\phi)=\phi(n=m)(\beta(n))^2 \qquad (m,n \in Z) \qquad (10.2)$$

and

$$|\phi(k)|(\beta(m+k))^2 \leqslant |f_{m+k}^*(M_\phi f_m)| \leqslant \|M_\phi\|\beta(m)\beta(m+k)$$

Hence

$$\beta(m+k)(\beta(m))^{-1} \leqslant |w|^{k+1}\|M_\phi\|$$

and from (1.2) and theorem 1-(iii), we obtain

$$\| M_z^k \| \leq |w|^k \| wM_\phi \| \quad (k \geq 0) \tag{11.2}$$

Taking k-th roots and letting $k \to \infty$, gives $r(T) \leq |w|$. By circular symmetry of $\sigma(T)$ it follows that $|w| > r(T)$. As $r(T^{-1}) = r(T)^{-1}$ it is deduced that $|w| \geq r(T^{-1})^{-1}$ and by circular symmetry again the result is concluded.

(ii) Firstly we prove that if $\phi \in L_p^\infty(\beta)$ then

$$|\phi(w)| \leq \|M_\phi\| \tag{12.2}$$

for all $w \in \sigma(T)$. Let w be with $|w| < r(T) = R$. From (4.2) it follows that $|\phi(n-m)|\beta(n) \leq \|M_\phi\|\beta(m)$ for all $n,m \geq 0$. If we divide by $\beta(n)$, and let $n=m+k$, and take infimum on k, from theorem 1 it follows that

$$|\phi(k)| \leq \| M_\phi \| \, \|T^k\|^{-1} \tag{13.2}$$

As $r(T) = \inf_{n \geq 0} \|T^n\|^{1/n}$, it follows that

$$|w|^k (\|T^k\|^{-1}) = (|w| \, \|T^k\|^{-1/k})^k \leq (|w| \, r(T)^{-1})^k \tag{14.2}$$

The expressions (13.2) and (14.2) imply that ϕ converges in the disc $|w| < r(T)$. As $r(T^{-1}) = r(T)^{-1}$, we obtain the convergence of ϕ in $|w| > r(T^{-1})^{-1}$. Now let $w = r(T)\exp(it)$, $t \in [0, 2\pi]$. Let us consider the sequence $\{\lambda_j\}_{j \in N}$ where λ_j is the functional of evaluation at $w_j = (r(T) - j^{-1})\exp(it)$. Note that λ_j is continuous on $L_p^\infty(\beta)$ because $|\phi(v)| \leq \|M_\phi\|$ when $|v| < r(T)$. By compacity of the structure space, [2], p.222, there exists a functional $\lambda \in K(L_p^\infty(\beta))$ and a subsequence $\{\lambda_{n_k}\}$ which converges to λ. Thus $\lambda(z) = \lim_{k\to\infty} \lambda_{n_k}(z) = w$. Thus $\lambda(z) = w$ and $\lambda_w(f) = \lim_{n,m\to\infty} (S_{n,m}(f))(w) = f(w)$. Moreover, if $\lambda_1, \lambda_2 \in K(L_p^\infty(\beta))$ satisfy $\lambda_1(z) = \lambda_2(z) = w$, by continuity $\lambda_1(f) = \lambda_2(f) = f(w)$. In consequence ϕ converges in $|w| \leq r(T)$ and a fortiori in $\sigma(T)$. Moreover, since $L_p^\infty(\beta)$ is a commutative Banach algebra with unit, it follows that $\|\lambda_w\| = 1$, for all $w \in \sigma(T)$ and for all $\lambda \in K(L_p^\infty(\beta))$ one has $\lambda(z)=w$. Thus (12.2) is proved.

By [2], p.223, if $f \in L_p^\infty(\beta)$ satisfies the hypothesis, it follows that

$$\sigma(f)=\{\lambda(f);\lambda \in K(L_p^\infty(\beta))\}= \{f(w);\ w \in \sigma(T)\}$$

because $\lambda(f)=f(w)$ when $\lambda(z)=w$ and for all $\lambda \in K(L_p^\infty(\beta))$, $\lambda(z) \in \sigma(T)$.

(iii) If $f \in L_p^\infty(\beta)$ and (6.2) is verified, given $\epsilon>0$, let n_o be a positive integer such that

$$\left\| \sum_{k=-n}^{m} f(k)T^k \right\| < \epsilon, \quad \text{if } m,n \geqslant n_o$$

From (12.2) it follows that

$$\sup_{w \in \sigma(T)} \left| \sum_{k=-n}^{m} f(k)w^k \right| < \epsilon\ , \text{ if } m,n \geqslant n_o$$

Hence the result is proved.

(iv) If $1 < p < \infty$ then the conjugate space of $L_p(b)$ is $L_q(b)$ where $1/p+1/q=1$, and for the case $p=1$, its conjugate space is the space $L_\infty(\beta)$ introduced in section 1. Let w be a complex number with $w \in \sigma(T)$ and let $\{f_n\}$ be the natural basis of $L_p(\beta)$ and let $\{f_n^*\}$ be the natural basis in its conjugate space. From the expression $\lambda_w(f_n)=w^n$, $n \in Z$, it follows that w is a bounded point evaluation on $L_p(\beta)$ if and only if there exists $g_w \in L_q(\beta)$ such that

$$w^n = \lambda_w(f_n)=g_w(n)f_n^*(f_n)=g_w(n)(\beta(n))^2;\ g_w(n)=w^n(\beta(n))^{-2}$$

where $1/p + 1/q = 1$ if $p>1$, and $q=\infty$ if $p=1$. Thus (iv) is proved.

(v) From theorem 1 it follows that $\|T^n\| \geqslant \beta(n)$ for all $n \in Z$. Moreover $\|T\|^n \geqslant \|T^n\|$ and $\|T^{-1}\|^n \geqslant \|T^{-n}\|$ for all $n \geqslant 0$. From (iv) the result is concluded.

(vi) It is clear that the Laurent series f, ϕ and ϕf converge absolutely at w to the values $\lambda_w(f), \lambda_w(\phi)$ and $\lambda_w(\phi f)$ respectively. From the classical theorem of Mertens the result is concluded.

If p=1 the result of theorem 3-(v) is not true as is showed in the following example.

Example 1. Let T be defined on $\ell_1(Z)$ by the expression $Te_n=e_{n+1}$ for all $n \in Z$. From theorem 1, it follows that $1=\|T\|=\|T^{-1}\|=\|T^n\|$ for all $n \in Z$. Moreover, it is clear that the sequence (8.2) is constant and w=1 is a bounded point evaluation from theorem 3-(iv).

If T is an unicellular operator then T has hyperinvariant subspace and every invariant subspace is hyperinvariant, [7]. Nikolskii, [6], and more recently B.S. Yadav and S. Chatterjee, [16], have found sufficient conditions for a weighted shift operator to be unicellular. In the following we prove that every bounded point evaluation generates a lot of invariant subspaces which are not linearly ordered by inclusion. We give neccesary conditions for unicellularity of an invertible bilateral shift on $\ell_p(Z)$.

Proposition 1. Let T be an invertible bilateral weighted shift on $\ell_p(Z)$ with $r(T^{-1})^{-1} < r(T)$, and let

$$s_1(T)=\limsup_{n\to\infty} (\beta(-n))^{-1/n}, \quad s_2(T)=\liminf_{n\to\infty} (\beta(n))^{1/n} \quad (15.2)$$

then it follows

(i) If p>1 and $s_1(T)<s_2(T)$ then T has bounded points evaluation and T is not unicelular.

(ii) If p=1 and w lies in the annulus $s_1(T) \leq |z| \leq s_2(T)$ then w is a bounded point evaluation and T is not unicellular.

Proof. Let w be a bounded point evaluation on $L_p(\beta)$ for $1 \leq p<\infty$, and let S(w) be the set of all f in $L_p(\beta)$ such that $\lambda_w(f)=f(w)=0$. As $S(w)=\mathrm{Ker}\,\lambda_w$, it is closed. We prove that S(w) is an invariant subspace of T. Let f be in S(w), then $Tf=M_zf=zf$ lies in S(w) because $\lambda_w(zf)=\lambda_w(z)\lambda_w(f)$ by theorem 3-(vi). Moreover, if w_1, w_2 are bounded points evaluation on $L_p(\beta)$ where $|w_1|=|w_2|$ and $w_1\neq w_2$, then the polynomials $g_i=z-w_i$, $i=1,2$ verify $g_i \in S(w_i)-S(w_j)$, for $i\neq j$, because

$$\lambda_{w_i}(g_j)=w_j-w_i \qquad (1 \leqslant i,j \leqslant 2)$$

Thus the lattice of invariant subspaces of T is not linearly ordered by inlcusion and T is not unicellular. (i) From theorem 3-(iv), w is a bounded point evaluation on $L_p(\beta)$ if p>1 and $s_1(T)<|w|<s_2(T)$. (ii) If p=1, then from (8.2), every point w such that $s_1(T) \leqslant |w| \leqslant s_2(T)$, is a bounded point evaluation. From here the result is established.

If $\|T^{-1}\|=\|T\|$, or more generaly if $r(T^{-1})^{-1}=r(T)$. and p>1, then proposition 1 is not sufficient to generate hyperinvariant subspaces. In the following results sufficient conditions to obtain hyperinvariant subspaces in these cases are given. Without loss of generality we suppose that $\|T\|=1$.

Theorem 4. Let $T=M_z$ on $L_p(\beta)$ an invertible bilateral weighted shift operator with $1 \leqslant p<\infty$. If $\|T\|=1$ and T satisfies

$$\sum_{n\in Z}(1-w_n)<\infty \tag{16.2}$$

then T admits a nontrivial hyperinvariant subspace.

Proof. Let T^* be the adjoint operator of T. It is clear that T^* is defined on $L_q(\beta)$ by the expression

$$T^*e_n = w_{n-1}e_{n-1}$$

for all $n\in Z$. Thus it follows that $(T^*)^n(e_o)=(w_{-1}w_{-2}\cdots w_{-n})e_{-n}$ for all $n \geqslant o$ and

$$\|(T^*)^n(e_o)\|= w_{-1}\cdots w_{-n} \qquad (n \geqslant 0) \tag{17.2}$$

Moreover it follows that

$$T^n(e_o)=(w_o\cdots w_{n-1})e_n \qquad (n\geqslant 0) \tag{18.2}$$

$$\|T^n\| \leqslant 1 \qquad (n\geqslant 0) \tag{19.2}$$

From the hypothesis (16.2) and theorem 15.5, [13], it follows that the infinite products

$$\prod_{n\geqslant 0} w_n > 0 \ ; \quad \prod_{n\geqslant 0} w_{-n}>0 \tag{20.2}$$

are convergent to positive limites. Thus we obtain

$$\lim_{n\to\infty} T^n(e_o)\neq 0 \ , \quad \lim_{n\to\infty}(T^*)^n(e_o)\neq 0 \tag{21.2}$$

From (19.2) and (21.2) and theorem 6.21, [12], the result is concluded.

Corollary 1. Let $T=M_z$ be an invertible bilateral weighted shift operator on $L_p(\beta)$ which satisfies the condition $\|T\|=\|T^{-1}\|^{-1}=1$, then T admits a nontrivial hyperinvariant subspace.

Proof. From the hypothesis $\|T\|=1$ it follows that $w_n \leqslant 1$ for all $n\in Z$. The hypothesis $\|T^{-1}\|=1$, implies

$$\|T^{-1}\| = \sup_{n\in Z}(w_n)^{-1}=1$$

Thus we obtain $w_n=1$ for all integer n. From theorem 4 the result is proved.

The following example shows that there are operators which satisfy the hypothesis of theorem 4 and they do not satisfy the hypothesis of corollary 1.

Example 2. Let $T=M_z$ on $L_p(\beta)$ where $1\leqslant p<\infty$, $w_{-n}=1$ for all $n\geqslant 1$ and $w_n=1-(n+1)^{-2}$ for all $n\geqslant 0$. Then it follows that

$$\|T\|=\sup_{n\in Z} w_n=1 \quad ; \quad \|T^{-1}\|=\sup_{n\in Z}(w_n)^{-1}=(w_1)^{-1}=3/4$$

$$\sum_{n\in Z}(1-w_n)=\sum_{n\geqslant 0}(n+1)^{-2}$$

Thus T satisfies (16.2) and does not satisfy the condition of corollary 1.

<u>Corollary 2.</u> Let $T=M_z$ be an invertible bilateral weighted shift on $L_p(\beta)$ with $1 \leqslant p < \infty$ and $w_n=c>0$ for all $n \in Z$. Then T admits a nontrivial hyperinvariant subspace.

<u>Proof.</u> Let V be the operator $V=T/\|T\|$, then V is the bilateral shift defined by $Ve_n=e_{n+1}$ for all $n \in Z$. Now theorem 4 proves the existence of hyperinvariant subspaces for V and a fortiori for T.

Note that an operator which satisfies the hypothesis of corollary 2 verifies $s_1(T)=s_2(T)$. Moreover if p=1, from proposition 1-(ii) the circle $|z|=s_1(T)=1$ is a set of bounded points evaluation and from theorem 3-(v), for the case p>1, these points are not bounded points evaluation and thus the existence of hyperinvariant subspaces for T is not insured.

<u>Theorem 5.</u> Let $T=M_z$ be an invertible bilateral weighted shift operator on $L_p(\beta)$ where $1 \leqslant p<\infty$ $\|T\|=1$. Suppose that T satisfies the following conditions

$$w_{-n} \leqslant w_{-(n+1)}, \quad \text{for all } n \geqslant 1 \tag{22.2}$$

$$\sum_{n\geqslant 0}(1-w_n)< + \infty \tag{23.2}$$

$$\sum_{n\geqslant 0}(1+n^2)^{-1}\log(w_{-1}\cdots w_{-n})^{-1}<+\infty \tag{24.2}$$

Then T admits a nontrivial hyperinvariant subspace.

<u>Proof.</u> From the hypothesis (23.2) and theorem 15.5, [13], it fo-

llows that

$$\lim_{n\to\infty} \|T^n(e_o)\| = \lim_{n\to\infty} \|w_o \ldots w_{n-1})e_{-n}\| = \lim_{n\to\infty}(w_o \ldots w_{n-1}) > 0 \quad (25.2)$$

Let $\rho_n = \|(T^{-n})(e_o)\| = \|(w_{-1} \ldots w_{-n})^{-1} e_n\| = (w_{-1} \ldots w_{-n})^{-1}$. We prove that the sequence $\{\rho_n\}_{n \geqslant 0}$ verifies

$$\rho_n \leqslant \rho_{n+1}, \ (n \geqslant 0) \quad (26.2)$$

$$\rho_{m+n} \leqslant \rho_m \rho_n, (n \geqslant 0, \ m \geqslant 0) \quad (27.2)$$

$$\sum_{n \geqslant 0} (1+n^2)^{-1} \log \rho_n < +\infty \quad (28.2)$$

From the expression $w_n \leqslant \|T\| = 1$, it follows that

$$\rho_n = (w_{-1} \ldots w_{-n}) \leqslant (w_{-1} \ldots w_{-n-1})^{-1} = \rho_{n+1}$$

Thus (26.2) is verified. The hypothesis (22.2) implies (27.2). In fact, it follows that

$$\rho_{m+n} = (w_{-1} \ldots w_{-m})^{-1} (w_{-m-1} \ldots w_{-m-n})^{-1} \leqslant (w_{-1} \ldots w_{-m})^{-1} (w_{-1} \ldots w_{-n}) = \rho_m \rho_n$$

The hypothesis (28.2) and the corollary of theorem 2, [1], besi des of conditions (25.2)-(28.2) imply the existence of hyperinvariant (nontrivial) subspaces for T.

There are operators which satisfy $r(T) = r(T^{-1})$ and the hypo thesis of theorem 5 and do not verify the hypothesis of theorem 4.

<u>Example 3.</u> Let $T = M_z$ on $L_p(\beta)$, $1 \leqslant p < \infty$ defined by the weight sequence $\{w_n\}_{n \in Z}$ where $w_n = 1$ for all $n \geqslant 0$ and $w_{-n} = 1-(n+1)^{-1}$ if $n \geqslant 1$. Then it follows that

$$\|T^n\| = \sup_{k \in Z}(w_k \ldots w_{k+n-1}) = 1; \ \|T^{-n}\| = \sup_{k \in Z}(w_k \ldots w_{k+n-1})^{-1} = n+1.$$

Thus $r(T)=1$, $r(T^{-1})=\lim_{n\to\infty}(n+1)^{1/n}=1$. As $w_{-n}\leqslant w_{-(n+1)}$ for all $n\geqslant 1$ by construction and $(w_{-1}\dots w_{-n})^{-1}=n+1$, it follows that

$$\sum_{n\geqslant 0}(1-w_n)=0,\quad \sum_{n\in Z}(1-w_n)=\sum_{n\geqslant 1}(1-w_{-n})=\sum_{n\geqslant 1}(n+1)^{-1}=+\infty$$

$$\sum_{n\geqslant 0}(1+n^2)^{-1}\log(n+1)<+\infty$$

Thus T satisfies the hypothesis of theorem 5 but it does not satisfy the hypothesis of theorem 4.

3. Analytic-spectral structure of the commutant of unilateral weighted shifts.

Let T be an injective unilateral weighted shift operator defined by (1.1), where I is the set of positive integers, I=N, and $\{e_n\}_{n\in N}$ is the natural basis of $\ell_p(N)$. In analogous way to the proof of theorem 1 and 2, in section 2, the follwing is easily established and we omit its proof.

<u>Theorem 6.</u> Let T be an operator on $\ell_p(N)$ defined by (1.1) with I=N, then

(i) T is an injective unilateral weighted shift on $\ell_p(N)$ if and only if T shifts some basis of $\ell_p(N)$, $1\leqslant p<\infty$.

(ii) Every injective unilateral weighted shift operator T on $\ell_p(N)$ can be represented as the operator M_z acting on $H_p(\beta)$, for a suitable β. The relation between $\{w_n\}_{n\geqslant 0}$ and β is given by the equations

$$\beta(0)=1,\quad \beta(n)=w_o\dots w_{n-1}\quad (n>0)$$

$$w_n=\beta(n+1)/\beta(n)$$

(iii) M_z is bounded if and only if $\{w_n\}_{n\geqslant 0}$ is bounded, and in this case

$$\|M_z^n\| = \sup_{k\geq 0} \beta(n+k)/\beta(k) \qquad (n \geq 0)$$

Moreover the spectrum of T is the disc $|z| \leq r(T)$.

The commutant of T will be represented by a certain algebra of formal power series, in the same way to the representation of the commutant of a bilateral weighted shift on $\ell_p(Z)$. Let us consider the multiplication of formal power series, h=fg

$$\left(\sum_{n\geq 0} f(n)z^n\right)\left(\sum_{n\geq 0} g(n)z^n\right) = \sum_{n\geq 0} h(n)z^n$$

where

$$h(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

For p fixed, with $1 \leq p<\infty$, let $H_p(\beta)$ be the space defined in the introduction, and let $H_p^\infty(\beta)$ be the set of formal power series $\phi(z) = \sum_{n\geq 0} \phi(n)z^n$, such that

$$\phi H_p(\beta) \subset H_p(\beta)$$

If $\phi \in H_p^\infty(\beta)$ then the product ϕf is defined and lies in $H_p(\beta)$ for all f in $H_p(\beta)$. It is clear that $H_p^\infty(\beta) \subset H_p(\beta)$. If $\phi \in H_p^\infty(\beta)$ then the linear transformation of multiplication by ϕ on $H(\beta)$ will be denoted by M_ϕ. The following results allow us to identify the commutant of T as $H_p^\infty(\beta)$. We omit its proof because it is analogous to the proof of [15], when p=2.

Proposition 2.

(i) If ϕ lies in $H_p^\infty(\beta)$ then M_ϕ is a bounded linear transformation on $H_p(\beta)$.

(ii) If ϕ, ψ belong to $H_p^\infty(\beta)$, then the product $\phi\psi$ is defined and is an element of $H_p^\infty(\beta)$ and $M_\phi M_\psi = M_{\phi\psi}$.

(iii) If B is an operator on $H_p(\beta)$ which commutes with M_z then $B=M_\phi$, for some ϕ in $H_p^\infty(\beta)$.

(iv) With the norm $\|\phi\|=\|M_\phi\|$, the space $H_p^\infty(\beta)$ is a commutative Banach algebra with unit.

In the following we identify the commutant of T with $H_p^\infty(\beta)$, and all operators which arise will be bounded.

Definition 2. If w is a complex number then λ_w denotes the functional of evaluation at w defined on polynomials by $\lambda_w(p)=p(w)$; w is said to be a bounded point evaluation on $H_p(\beta)$ if the functional λ_w extends to a bounded linear functional on $H_p(\beta)$.

Given an injective unilateral weighted shift operator T on $\ell_p(N)$, $1 \leqslant p<\infty$ and represented by M_z on the space $H_p(\beta)$, we denote

$$s(T)=\lim_{n\to\infty}(\beta(n))^{1/n} \tag{1.3}$$

We denote by $K(H_p(\beta))$ the structure space of the commutative Banach algebra $H_p^\infty(\beta)$. The following theorem shows the analytic - spectral structure of the commutant $H_p^\infty(\beta)$.

Theorem 7. Let $T=M_z$ on $H_p(\beta)$ where $1 \leqslant p<\infty$ and $r(T)>0$, then it follows that

(i) If $|w|<r(T)$ and $\phi\in H_p(\beta)$ then the power series ϕ converges at w and

$$|\phi(w)| \leqslant \|M_\phi\| \tag{2.3}$$

(ii) If f belongs to $H_p^\infty(\beta)$ and $S_n(f)=\sum_{k=0}^{n} f(k)z^k$, then $\{S_n(f)\}_{n\geqslant 0}$ is pointwise convergent to the function which defines f, in $\sigma(T)$. Moreover, f is continuous in $\sigma(T)$ and analytic in $|z|<r(T)$.

(iii) If f belongs to $H_p^\infty(\beta)$ and $\{S_n(f)\}_{n\geqslant 0}$ converges to f in

$H_p(\beta)$, then

$$\sigma(f)=\{f(w);\ |w|\leqslant r(T)\} \tag{3.3}$$

<u>Proof.</u> (i) In analogous way to the proof of theorem 3-(ii), it follows that

$$|\phi(k)|\leqslant \|M_\phi\|\,\|T^k\|^{-1} \tag{4.3}$$

From the properties of the spectral radius, 4 , one has $r(T)=\inf_{n\geqslant 0}\|T^n\|^{1/n}$. Thus, if w is a complex number which satisfies $|w|<r(T)$ it follows that

$$|w|^k(\|T^k\|^{-1}) \leqslant (|w|\,r(T)^{-1})^k \tag{5.3}$$

From (4.3) and (5.3) it results that ϕ converges in the disc $|w|<r(T)$. Moreover since $H_p^\infty(\beta)$ is a commutative Banach algebra with unit, and the functional of evaluation at w has norm one, $\|\lambda_w\|=1$, we obtain

$$|\lambda_w(\phi)|=|\phi(w)| \leqslant \|M_\phi\|$$

(i) From (i), f defines an analytic function in the disc $|z| \leqslant r(T)$ and from Gel'fand's theorem, [2], p.223, f is a continuous function in $\sigma(T)=\{z;|z| \leqslant r(T)\}$.

(iii) It is analogous to the proof of theorem 3-(ii).

The following result continues the study of the analytic structure of $H_p^\infty(\beta)$.

<u>Theorem 8.</u> Let $T=M_z$ on $H_p(\beta)$ where $1 \leqslant p<\infty$, then it follows that

(i) If w is a complex number $|w|<s(T)$, then w is a bounded point evaluation on $H_p(\beta)$.

(ii) Let $f\in H_p(\beta)$. If w is a bounded point evaluation on $H_p(\beta)$, then the power series which defines f converges absolutely at w and $\lambda_w(f)=\sum_{n\geq 0} f(n)w^n$. Moreover if w is a bounded point evaluation on $H_p(\beta)$, $\phi \in H_p^\infty(\beta)$ and $f \in H_p(\beta)$, then $\lambda_w(\phi f) = \lambda_w(\phi)\lambda_w(f)$.

Proof. (i) Let $\{f_n\}$ be the natural basis of $H_p(\beta)$ and let $\{f_n^*\}$ be the natural basis in its conjugate space $H_q(\beta)$, where $1/p+1/q=1$ if $p>1$, and $q=\infty$ if $p=1$. From the expression $\lambda_w(f_n)=w^n$, it follows that w is a bounded point evaluation on $H_p(\beta)$ if and only if there exists $g_w \in H_q(\beta)$ such that

$$w^n = \lambda_w(f_n)=g_w(n)f_n^*(f_n)=g_w(n)(\beta(n))^2,\quad g_w(n)=w^n(\beta(n))^{-2}$$

Thus w is a bounded point evaluation if and only if

$$\sum_{n\geq 0} |w|^{nq}(\beta(n))^{-q}<+\infty, \text{ if } p>1 \tag{6.3}$$

$$\{|w|^n(\beta(n))^{-1}\}_{n\geq 0} \text{ is bounded, when } p=1 \tag{7.3}$$

The result is a consequence of the expressions (6.3), (7.3) and (1.3).

(ii) Let $S_n(f)=\sum_{k=0}^{n} f(k)z^k$, and let $f \in H_p(\beta)$. As $\{S_n(f)\}_{n\geq 0}$ converges to f in $H_p(\beta)$ with respect to $\|\|_p$ and λ_w is bounded on $H_p(\beta)$, it follows that $\lambda_w(f)=\lim_{n\to\infty} \lambda_w(S_n(f))$. Moreover from the expression $p(w)=\lambda_w(p)$ for polynomials we obtain

$$\lambda_w(f)=\lim_{n\to\infty} \sum_{k=0}^{n} f(k)w^k=\sum_{n\geq 0} f(n)w^n \tag{8.3}$$

By circular symmetry of the spectrum, we have convergence at w for all f in $H_p(\beta)$, and as $f \in H_p(\beta)$ if and only if $|f| \in H_p(\beta)$, where $|f|$ is the power series $\sum_{n\geq 0}|f(n)|z^n$, it follows that the series (8.3) is absolutely convergent. From the expression (2.3), the power series f, ϕ and ϕf converge absolutely at w to the values $\lambda_w(f)$, $\lambda_w(\phi)$ and $\lambda_w(\phi f)$, respectively. Thus it follows that $\lambda_w(\phi f)=\lambda_w(f)\lambda_w(\phi)$.

Remark 1. If $p>1$ and $|w|=\|T\|$, then w is not a bounded point evaluation on $H_p(\beta)$ because $\|T\|^n \geq \|T^n\| \geq \beta(n)$, for all $n \geq 0$ and thus $\|T\|^n(\beta(n))^{-1} \geq 1$ for all $n \geq 0$. From the proof of theorem 2-(i) it follows that w is not a bounded point evaluation on $H_p(\beta)$. If $p=1$ and $|w|=\|T\|$, then w can be a bounded point evaluation on $H_1(\beta)$. For example if $w_n=c>0$ for all $n \geq 0$ then $\beta(n)=\|T\|^n = c^n$, for all $n \geq 0$. From (7.3) it follows that w is a bounded point evaluation on $H_1(\beta)$.

It is well known that the only operators which are unicelular in finite-dimensional spaces have a spectrum reduced to a point. C. Foias and J.P. Williams [8], have given an example of an unicellular operator with more than one point in its spectrum. In the context of weighted shift operators several authors, [11], and [16], have found sufficient conditions for unicellularity.

We omit the proof of the following corollary because it is easy from the proof of the proposition 1.

Corollary 3. Let $T=M_z$ on $H_p(\beta)$, where $1 \leq p<\infty$. If $s(T)>0$ then T is not unicelular.

If T is an injective unilateral weighted shift operator on $\ell_p(N)$, where $1 \leq p<\infty$ and $r(T)=0$, that is, T is quainilpotent, then the analytic-spectral structure of the commutant $H_p^\infty(\beta)$ obtained in theorems 7, 8 and the corollary 3 where $r(T)>0$ is not available. In the following result this question is studied.

Theorem 9. Let $T=M_z$ on $H_p(\beta)$ an injective unilateral weighted shift operator where $1 \leq p<\infty$, and let us suposse that $r(T)=0$. Then it follows that

(i) If $f(z)=\sum_{n\geq 0} f(n)z^n$ is a power series with radius of convergence $r>0$, then $f\in H_p^\infty(\beta)$ and $S_n(f) \to f$ with respect to the norm of $H_p^\infty(\beta)$.

(ii) There exists a power series $f(z)=\sum_{n\geq 0} f(n)z^n$ with radius of convergence r=0 and $f \in H_p^\infty(\beta)$.

(iii) If $f \in H_p^\infty(\beta)$ and $S_n(f) \to f$ in $H_p(\beta)$, then $\sigma(f)=\{f(0)\}$.

(iv) There exists a power series $f(z)=\sum_{n\geq 0} f(n)z^n$, with radius of convergence r=0 and $f \in H_p^\infty(\beta)$.

<u>Proof.</u> (i) Let $f(z)=\sum_{n\geq 0} f(n)z^n$, with $r^{-1}=\lim_{n\to\infty}\sup|f(n)|^{1/n}<+\infty$, then the operator $A=\sum_{n\geq 0} f(n)T^n$ is bounded because the series $\sum_{n\geq 0}|f(n)|\,\|T^n\|$ is convergent

$$\lim_{n\to\infty}\sup\{|f(n)|\,\|T^n\|\}^{1/n}=\lim_{n\to\infty}\sup|f(n)|^{1/n}\lim_{n\to\infty}\|T^n\|^{1/n}=0$$

As AT=TA, from proposition 2-(iii) there exists $g\in H_p^\infty(\beta)$ such that $A=M_g$ on $H_p(\beta)$, where $g(z)=A(f_o)(z)=\sum_{n\geq 0} f(n)M_z^n f_o(z)=\sum_{n\geq 0} f(n)z^n$. It is clear from the convergence of $\sum_{n\geq 0}|f(n)|\,\|T^n\|$ that $f=\lim_{n\to\infty} S_n(f)$ in $H_p^\infty(\beta)$. This proves (i).

(ii) Let f be $f(z)=\sum_{n\geq 0}(\beta(n))^{-1}z^n$, then as $s(T)\leqslant r(T)=0$, it follows that this power series has radius of convergence r=0. Moreover $f \notin H_p(\beta)$ because

$$\sum_{n\geq 0}|f(n)|^p(\beta(n))^p=\sum_{n\geq 0}1=+\infty$$

(iii) The proof of theorem 3-(ii).

(iv) Let k be a real number with $0<k<1$, $f(n)=k^n/\|T^n\|$, for $n\geqslant 0$, then $f\in H_p^\infty(\beta)$ because the operator $\sum_{n\geq 0} f(n)\,T^n$ is well defined and commutes with T. Moreover the power series which defines f has radius of convergence r=0 because

$$\lim_{n\to\infty}\sup|f(n)|^{1/n} = k\lim_{n\to\infty}\sup\|T^n\|^{-1/n}=+\infty$$

Let $T=M_z$ on $H_p(\beta)$, for $1<p<\infty$, we recall that T is called strictly cyclic if there exists a vector $f\in H_p(\beta)$, such that $(H_p^\infty(\beta))f = H_p(\beta)$.

If $\{w_n\}_{n\geq 0}$ is monotonically non-increasing and $\sum_{n\geq 0}(\beta(n))^{-q}<\infty$, for $1<p<\infty$ then the operator $Te_n = w_n e_{n+1}, n\geq 0$ on $\ell_p(N)$ can be non-strictly cyclic as proved D.A. Herrero, [7], and H. Salas, [14]. The following theorem gives sufficient conditions for strictly cyclicity of such operators. Results of [9], obtained when p=2 are extended.

Theorem 10. Let $T=M_2$ on $H_p(\beta)$, an injective weighted shift operator on $\ell_p(N)$ where $1<p<\infty$ and suppose that

$$w_{n+1}\leq w_n,\ (n\geq 0)\ \text{and}\ \lim_{n\to\infty} w_n = 1 \qquad (9.3)$$

$$\sum_{n\geq 0}(\beta(n))^{-q}<\infty,\quad p^{-1}+q^{-1}=1 \qquad (10.3)$$

$$\sup_{n\geq 0}\left\{\frac{\beta(2n)}{\beta(n)}\right\}<\infty \qquad (11.3)$$

Then T is strictly cyclic.

Proof. As $w_{n+1}\leq w_n$, for $n\geq 0$, from [10], T is strictly cyclic if and only if,

$$\sup_{n\geq 0}\sum_{k=0}^{n}\left(\frac{\beta(n)}{\beta(k)\beta(n-k)}\right)^q \qquad (12.3)$$

The condition (12.3) will be satisfied if the sequences $\{S_{2n}\}_{n\geq 0}$ and $\{S_{2n+1}\}$ are bounded, where

$$S_j=\sum_{k=0}^{j}\left(\frac{\beta(j)}{\beta(k)\beta(j-k)}\right)^q \qquad (j\geq 0)$$

For $n\geq 0$ it follows that

$$S_{2n}=(\beta(2n))^q\sum_{k=0}^{2n}(\beta(k)\beta(2n-k))^{-q}\leq 2(\beta(2n))^q\sum_{k=0}^{n}(\beta(k)\beta(2n-k))^{-q}$$

As $w_j \leq 1$ for all $j \geq 0$, and $(\beta(2n-k))^{-q} \leq (\beta(n))^{-q}$, for $k=0,1,\dots,n$, it follows that

$$S_{2n} \leq 2\left(\frac{\beta(2n)}{\beta(n)}\right)^q \sum_{k=0}^{n} (\beta(k))^{-q} \qquad (13.3)$$

Analogously for $n \geq 0$ it follows that

$$S_{2n+1} = (\beta(2n+1))^q \sum_{k=0}^{2n+1} (\beta(k)\beta(2n+1-k))^{-q} \leq 2\left(\frac{\beta(2n+1)}{\beta(n+1)}\right)^q \sum_{k=0}^{n} (\beta(k))^{-q} \qquad (14.3)$$

Moreover as $\beta(2n+n)=w_{2n+1}\beta(2n+1)$, from (14.3) it follows that

$$S_{2n+1} = 2\left(\frac{\beta(2n+2)}{\beta(n+1)}\right)^q (w_{2n+1})^{-q} \sum_{k=0}^{n} (\beta(k))^{-q} \qquad (15.3)$$

From (9.3) the sequence $\{(w_{2n+1})^{-q}\}_{n \geq 0}$ is bounded and from (11.3) the sequence $\{\beta(2n)/\beta(n)\}_{n \geq 0}$ is bounded. From here and the expressions (10.3), (13.3) and (15.3), the result is proved.

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