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SOME RESULTS ABOUT ABSOLUTE SUMMABILITY OF OPERATORS IN BANACH SPACES

by

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ABSTRACT

In order to study the absolute summability of an operator T we consider the set $S_T = \{\{x_n\} \mid \Sigma\|Tx_n\| < \infty\}$. *It is well known that an operator T in a Hilbert space is nuclear if and only if* S_T *contains an orthonormal basis and it is natural to ask under wich conditions two orthonormal basis define the same left ideal of nuclear operators. Using results about* S_T *we solve this problem in the more general context of Banach spaces.*

For general concepts and notations we refer to [2], [3], in all the following E, F, E_1, E_2 will denote normed spaces over the field Φ of real or complex numbers.

$$\ell_1(E) = \left\{\{x_n\} \subset E \mid \Sigma\|x_n\| < \infty\right\}, \quad \|\{x_n\}\|_1 = \Sigma\|x_n\|$$

$$\ell_\infty(E) = \left\{\{x_n\} \subset E \mid \text{Sup}\{\|x_n\|\} < \infty\right\}, \quad \|\{x_n\}\| = \text{Sup}_n\{\|x_n\|\}$$

If by E^* we denote the dual of the Banach space E we have that

$$[\ell_1(E)]^* = \ell_\infty(E^*).$$

Given a linear (not necessarily bounded) operator $T: E \to F$ we define S_T as

$$S_T = \left\{ \{x_n\} \subset E \mid \Sigma \|Tx_n\| < \infty \right\}, \ \|\{x_n\}\|_T = \Sigma \|Tx_n\|.$$

Every map between sets $\phi : A \to B$ induces a map: $A^N \to B^N$, $\{a_n\} \to \{\phi a_n\}$ which we will denote by $\mathring{\phi}$; we will also write $\mathring{\phi}$ for a restriction of ϕ to any subset of A^N.

<u>Theorem 1.</u> If $T: E_1 \to E_2$ is a linear bounded operator, $\mathring{T}: \ell_1(E_1) \to \ell_1(E_2$ is also bounded with $\|\mathring{T}\| = \|T\|$.

<u>Theorem 2.</u> For every linear operator $T: E \to F$, where F is a Banach space, $\mathring{T}: S_T \to \ell_1(\overline{TE})$ is a completion of S_T.

<u>Theorem 3:</u> If ΣM_p is a decomposition of E, and $T: E \to F$ is a linear operator where F is a Banach space the following are equivalent.

1) $\overline{TE} = \Sigma \overline{TM}_p$ is a decomposition of $\overline{TE}$.

2) $\ell_1(\overline{TE}) = \Sigma \ell_1(\overline{TM}_p)$ is a decomposition of $\ell_1(\overline{TE})$.

3) $S_T = \Sigma S_{T|M_p}$ is a decomposition of S_T with a uniformly bounded associated family of projections.

Proof:

In this proof we will make use of the following result ([3], VII, 2.8):

"Let $\{P_n\}$ be a sequence of mutually orthogonal continuous projections of the Banach space X, such that $\overline{sp} \overset{\infty}{\underset{n=1}{\cup}} P_n(X) = X$. Then $\Sigma P_n(X)$ is a Schauder decomposition of X if and only if there exists a constant $K \geq 1$ such that $\sup_n \| \sum_{i \leq n} P_i \| \leq K$".

1) ⇒ 2): If $\{P_p\}$ is the associated family of projections and K the constant of the decomposition $\Sigma \overline{TM}_p$, we have that $\{\mathring{P}_p\}$ is a mutually orthogonal family of projections and by Theorem 1

$$\sup_p \{\| \sum_{i \leq p} \mathring{P}_i \|\} \leq K.$$

To show the remaining condition let us consider

$$H^r = \left\{ \{z_n\} \in \ell_1(\overline{TE}) \,|\, z_n = 0 \quad \forall n \neq r \right\}.$$

ΣH^r is a decomposition of $\ell_1(\overline{TE})$, and $\Sigma \overline{TM}_p$ being a decomposition $\overline{TE}$ we have

$$H^r \subset \overline{sp} \overset{\infty}{\underset{1}{\cup}} \mathring{P}_p[\ell_1(\overline{TE})]$$

3) ⇒ 2) According to Theorem 2 $(\ell_1(\overline{TE}), \mathring{T})$ and $(\ell_1(\overline{TM}_p), \mathring{T})$ are completions of S_T and $S_{T|M_p}$ respectively, so that each $\mathring{P}_p$ admits a unique continuous extension π_p and, as it is easily seen, the family $\{\pi_p\}$ verifies the conditions of the above results.

The rest of the proof is straighforward.

It is natural to ask if the induced decompositions of this theorem keep some of the properties of the original decompositions; the following result shows that this is in fact the case for monotone and unconditional decompositions.

<u>Theorem 4.</u> In the same context of the previous theorem

a) $\Sigma \overline{TM}_p$ monotone ⇔ $\Sigma \ell_1(\overline{TM}_p)$ monotone ⇔ $\Sigma S_{T|M_p}$ monotone.

b) $\Sigma \overline{TM}_p$ unconditional $\Leftrightarrow$ $\Sigma \ell_1(\overline{TM}_p)$ unconditional

$\Leftrightarrow$ $\Sigma S_{T|M_p}$ unconditional.

Proof:

Suppose $\Sigma \overline{TM}_p$ is unconditional and let

$\{z_n\} \in \ell_1(\overline{TE})$

$\xi \in [\ell_1(\overline{TE})]^*$, i.e. $\xi = \{\psi_n\}$, ψ_n $\overline{TE}^*$, $\|\psi_n\| \leqslant K_1$ $\forall n$.

If we put $z_n^p = P_p z_n$ then $\langle z_n, \psi_n \rangle = \sum_p \langle z_n^p, \psi_n \rangle$ and for the double series $\Sigma \langle z^p, \psi_n \rangle$ we have

$$\left| \sum_{\substack{n \leqslant n' \\ p \leqslant p'}} \langle z_n^p, \psi_n \rangle \right| \leqslant \sum_{\substack{n \leqslant n' \\ p \leqslant p'}} \left| \langle z_n^p, \psi_n \rangle \right| =$$

$$= \sum_{\substack{n \leqslant n' \\ p \leqslant p'}} \langle \varepsilon_n^p z_n^p, \psi_n \rangle \leqslant \sum_{n \leqslant n'} \|\psi_n\| \, \| \sum_{p \leqslant p'} \varepsilon_n^p z_n^p \| \leqslant$$

$$\leqslant K_1 K_2 \|\{z_n\}\|_1 .$$

for convenient ε_n^p with $|\varepsilon_n^p| = 1$, K_2 being the unconditional constant of the decomposition $\Sigma \overline{TM}_p$. It follows that the double series and hence any of its subseries is absolutely convergent; then for every permutation ω of N, $\Sigma \ell_1(\overline{TM}_{\omega(p)})$ is a weak decomposition of $\ell_1(\overline{TE})$.

The other implications are easy to verify.

Given a basis $\{e_n\}$ of the normed space E, $C_{\{e_n\}}$ will denote the left ideal of operators

$$C_{\{e_n\}} = \{T \in \mathcal{B}(E) \,|\, \Sigma \|T e_n\| < \infty\}$$

Two basis will be called equivalent (written $\{e_n\}R\{f_n\}$) when

$$C_{\{e_n\}} = C_{\{f_n\}}$$

<u>Theorem 5.</u> Let $\{e_n\},\{f_n\}$ be basis of the Banach space E, and let $\alpha_n = \sum_m |<f_m, e_n^*>|$, $\beta_m = \sum_n |<e_n, f_m^*>|$ where $\{e_n^*\}$, $\{f_m^*\}$ denote the associated coordinate functionals of $\{e_n\}$, $\{f_m\}$ respect. Then:

$$\{e_n\}R\{f_m\} \Leftrightarrow \{\alpha_n\},\{\beta_m\} \in \ell_\infty$$

Proof:

Let us assume that $\{e_n\}R\{f_n\}$. Firstly we show that β_m exists.

Let $T = \sum \frac{f_m^* \otimes f_m}{m^2 \| f_m \|}$

clearly $T \in C_{\{f_m\}}$ and then $\{T_{e_n}\} \in \ell_1(\overline{TE})$.

On the other hand $T\, sp(f_m) = sp(f_m)$, and $\overline{TE} = \sum sp(f_m)$ is a decomposition of $\overline{TE}$, by Theorem 3 we have that $\ell_1(\overline{TE}) = \sum \ell_1(sp(T\, f_m))$ wich means that:

$$\{T_{e_n}\} = \sum_m \left\{ \frac{<e_n, f_m^*> f_m}{m^2 \| f_m \|} \right\}$$

thus for each, $m, \beta_m/m^2 = \left|\left| \left\{ \frac{<e_n, f_m^*> f_m}{m^2 \| f_m \|} \right\} \right|\right|_1 < \infty$

Now suppose that $\{\beta_m\} \notin \ell_\infty$ i.e. there exists $\{\beta_{m_r}\} \subset \{\beta_m\}$ with $\beta_{m_r} > r^2$, and consider the operator

$$T' = \sum_r \frac{f^*_{m_r} \otimes f_{m_r}}{r^2 \| f_{m_r} \|}$$

again we have $T' \in C_{\{f_n\}}$ and according to theorem 3

$\ell_1(\overline{T'E}) = \sum_r \ell_1(T' \; sp\{f_{m_r}\})$, it follows

$$\{T'e_n\} = \sum_r \left\{\frac{<e_n, f^*_{m_r}> f_{m_r}}{r^2 \| f_{m_r} \|}\right\}$$

but

$$\left|\left|\left\{\frac{<e_n, f^*_{m_r}> f_{m_r}}{r^2 \| f_{m_r} \|}\right\}\right|\right|_1 = \beta_{m_r} / r^2 > 1$$

Conversely let us assume that $\{\beta_m\} \in \ell_\infty$ and let

$T \in C_{\{f_m\}}$: $\sum_n \| T_{e_n} \| \leqslant \sum_n \| \sum_m <e_n, f^*_m> Tf_m \| \leqslant \sum_m \beta_m \| Tf_m \| < \infty$

The necessary condition of the last theorem expresses a rather negative result regarding perturbation of basis:

Let $\{e_n\}$ be an orthonormal basis of a Hilbert Space and let f such that $\sum |(f, e_n)| = \infty$. Then there is no basis $\{f_n\}$ such that $f \in \{f_n\}$ and $\| e_n - f_n \| < \infty$, otherways we would have $\{e_n\} R \{f_n\}$.

References.

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