STOCHASTICA Vol. IX Nº 2, 1985.

A SIMPLE PROOF OF UNIQUENESS FOR TORSION MODULES OVER PRINCIPAL IDEAL DOMAINS

J.L. García Roig

ABSTRACT

The aim of this note is to give an alternative proof of uniqueness for the decomposition of a finitely generated torsion module over a P.I.D. (= principal ideal domain) as a direct sum of indecomposable submodules.

Our proof tries to mimic as far as we can the standard procedures used when dealing with vector spaces.

For the sake of completeness we also include a proof of the existence theorem.

We recall that if M is a finitely generated torsion module over a P.I.D. R then $M=\oplus M_p$, where p runs through a complete set of representatives of prime (or irreducible) elements of R. The M_p are unique: in fact $M_p = \bigcup_{n\geq 1} (0:p^n)_M$. Moreover, if $Ann(M)=(p_1^{n_1}\cdots p_k^{n_k})$, then $M_p=0$ for $p \notin \{p_1,\ldots,p_k\}$, i.e., $M=\bigoplus_{i=1}^{k} M_{p_i}$.

Our interest now concerns each summand, i.e., we consider a finitely generated module annihilated by a power of an irreducible element p of R. For these we have

Existence Theorem. Let R be a P.I.D., p a prime element in R and $M \neq 0$ a finitely generated R-module annihilated by some power of p. Then we can write

$$M \cong R/p^{n_1}R \oplus \dots \oplus R/p^{n_k}R$$

(the isomorphism being of R-modules, or what amounts to the same thing, of R/I-modules, for any $I \subseteq \mathrm{Ann}(M)$), with $n_1 \geqslant \dots \geqslant n_k \geqslant 1$.

Proof. By induction on the number of generators. Let $x_1, \dots, x_m$ be a finite set of generators for M. We may assume without loss of generality that x_1 has maximal period, say p^n, $n>0$ (so that in particular $\mathrm{Ann}(M)=p^nR$). Consider the exact sequence of R-modules:

$$0 \to x_1R \to M \to \bar{M} \to 0$$

with $\bar{M} = M/x_1R$. This can be viewed as an exact sequence of R/p^nR-modules and as R/p^nR is an injective R/p^nR-module (see below), we have the splitting

$$M \cong x_1R \oplus \bar{M} \cong R/p^nR \oplus \bar{M}.$$

As $\bar{M}$ is generated by the cosets of $x_2, \dots, x_m$, induction may be applied.

R/p^nR is an injective R/p^nR-module: The ideals of $R/p^nR =: S$ are $\{p^iS \mid i=0,1,\dots,n\}$, so it is enough to see that any S-homomorphism $f: p^iS \to S$ has an extension to the whole of S. But f is defined by its effect on $\bar{p}^i$ (=coset of p^i in S). Let $f(\bar{p}^i) = \bar{q}$. As p^{n-i} annihilates $\bar{p}^i$ and therefore $\bar{q}$, we may write $\bar{q} = \bar{p}^i\bar{r}$ for some $\bar{r} \in S$. Defining now $\bar{f}: S \to S$ by $\bar{f}(\bar{1}) = \bar{r}$, we get an extension of f. #

Uniqueness Theorem. The sequence $n_1,\ldots,n_k$ appearing in the Existence Theorem is uniquely determined by M.

Proof. The Existence Theorem allows us to write $M=x_1R\oplus\ldots\oplus x_kR$, with $Ann(x_iR)=p^{n_i}R$, $n=n_1\geqslant\ldots\geqslant n_k\geqslant 1$. Suppose we also have $M=y_1R\oplus\ldots\oplus y_hR$ with $Ann(y_jR)=p^{m_j}R, m_1\geqslant\ldots\geqslant m_h\geqslant 1$. We must necessarily have $n=m_1$. Then $y_1=a_1x_1+\ldots+a_kx_k$, for some $a_i\in R$ and without loss of generality we may assume a_1 and p to be coprime - otherwise the period of y_1 would not be p^n - whence there exist r,s in R such that $1=a_1r+sp^n$. So $ry_1=x_1+ra_2x_2+\ldots+ra_kx_k$, i. e., $x_1=ry_1-ra_2x_2-\ldots-ra_kx_k$, and thus $y_1,x_2,\ldots,x_k$ generate M. Furthermore, if $b_1y_1+b_2x_2+\ldots+b_kx_k=0$, bearing in mind that $y_1=a_1x_1+\ldots+a_kx_k$ we conclude that $p^n|b_1$ so that $b_1y_1=0$, and consequently each summand b_jx_j, $j>1$, is zero. In other words, $M=y_1R\oplus x_2R\oplus\ldots\oplus x_kR$, and from this we infer

$$x_2R\oplus\ldots\oplus x_kR\cong y_2R\oplus\ldots\oplus y_hR.$$

Now we can apply induction to get uniqueness. #

We cannot proceed further, for the R-modules of type R/p^nR, p prime, are indecomposable, i.e., cannot be expressed as a direct sum of proper submodules. This is so because, as we have remarked before, the submodules are of the form p^iR/p^nR, with $0\leqslant i\leqslant n$.

References.

[1] LANG, S.: Algebra. Addison-Wesley, 1965.

[2] VAN DER WAERDEN, B. L.: Algebra II. (5. Auflage).Springer 1967.

Manuscript received in January 30, 1986.

CRUZ ROJA 26, 3er.4a.HOSPITALET BARCELONA. SPAIN.