NOTAS BREVES

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ORLICZ METRICS DERIVE FROM A SINGLE PROBABILISTIC METRIC

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ABSTRACT

It is shown that the same probabilistic metric as used by Schweizer and Sklar to obtain all L^p *space metrics can be used to derive the metrics of Orlicz spaces.*

Let (Ω, A, μ) be a probability space and let λ denote the Lebesgue measure on (R, B). Further let φ and ψ be a pair of complementary Young functions, i. e. $\varphi, \psi : R \to R_+$ are such that

$$\varphi(x) = \int_0^{|x|} h \, d\lambda \quad , \quad \psi(x) = \int_0^{|x|} h^* \, d\lambda \ ,$$

where $h, h^* : R \to R_+$ are right-continuous, increasing functions such that $\lim_{x \to +\infty} h(x) = +\infty$, $h(0) = h^*(0) = 0$ and $h^*(t) := \sup\{s : h(s) \leqslant t\}$. Let L^0 be the set of equivalence classes (with respect to equality μ-a.e.) of measurable functions $f : \Omega \to R$. Finally Let $L^{\varphi} = L^{\varphi}(\Omega, A, \mu) \subset L^0$ denote those functions f for which $|E(fg)| < +\infty$ for every measurable function $g : \Omega \to R$ such that $E(\psi \circ g) < +\infty$. Such a space is a Banach space, called an Orlicz space, under either the Orlicz or the Luxemburg norm, which are equivalent ([3]); we shall use

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the Luxemburg norm.

(1) $$||f||_{\varphi} := \inf\{k>0 : E[\varphi(|f|/k)] \leq 1\} \quad .$$

Schweizer and Sklar ([5]) have shown how to derive all L^p-metrics from a single probabilistic metric. On the other hand, one has

(a) $L^p \subset L^1$ $(p>1)$;

(b) every L^p space $(p>1)$ is an Orlicz space with $\varphi_p(x)=|x|^p/p$ and $||f||_p=(p^{1/p})\,||f||_{\varphi_p}$;

(c) $L^1 = \cup\{L^{\varphi} : \varphi$ is a Young function$\}$ ([4], 1.4.5).

In view of these facts and of the relevance of Orlicz spaces in the characterization of conditional expectations ([4]), it is entirely natural to ask whether the result just mentioned can be generalized so as to derive all Orlicz space distances from the same probabilistic metric as used by Schweizer and Sklar. The purpose of this note is to answer this question in the affirmative and to show that their result, for $p \in]1,+\infty[$, is a particular case of the main result below.

For the sake of completeness the preliminaries of [5] will be repeated here. A function of L° will be called a random variable (r.v.). For any pair (f,g) of r;v;'s and for any $x>0$ the set $\{\omega \in \Omega : d[f(\omega),g(\omega)]<x\}$ is measurable; here d denotes the usual distance on R. The function $F_{fg}: R_+ \to [0,1]$ defined by

$$F_{fg}(x) := \mu\{\omega\in\Omega : d[f(\omega),g(\omega)]<x\} \quad 1_{]0,+\infty]}(x)$$

is the (left-continuous) distribution function of the r.v. $d(f,g)$. In this way one defines a mapping $\mathcal{F}: L^\circ \times L^\circ \to \Delta^+$ (the space of left-continuous distribution functions F with $F(0)=0$; see [6]), by $\mathcal{F}(f,g) := F_{fg}$. The pair $(L^\circ, \mathcal{F})$ is a probabilistic metric space ([6]), viz. for all $f,g,h \in L^\circ$ one has

(i) $\mathcal{F}(f,g)=\varepsilon_\circ$ if and only if $f=g$;

(ii) $\mathcal{F}(f,g)=\mathcal{F}(g,f)$;

(iii) $\mathcal{F}(f,g) \geqslant \tau_m[\mathcal{F}(f,h),\mathcal{F}(h,g)]$,

where $\varepsilon_\circ:=1_{]0,+\infty]}$ and for any two distribution functions F,G in Δ^+ and for any $t>0$, τ_m is defined by

$$\tau_m(F,G)(t):=\sup_{x+y=t} \max\{0,F(x)+G(y)-1\} \quad .$$

For a proof of this, see [5] theorem 1.

One can now prove the main result.

<u>Theorem.</u> For any $f,g \in L^\varphi$ one has

$$(2) \qquad d_\varphi(f,g):= ||d(f,g)||_\varphi=\inf\{k>0: \int_{R_+} \varphi(x/k)\ dF_{fg}(x) \leqslant 1\}.$$

Proof. By recourse to the usual change of variable formula for integrals (see, for instance, [2] theorem 11.4) one has

$$E[\varphi(d(f,g)/k)] = \int_\Omega \varphi[d\{f(\omega),g(\omega)\}/k]\ d\mu(\omega) =$$

$$= \int_{R_+} \varphi(x/k)\ dF_{fg}(x)$$

and hence (2), on account of the definition (1). Q.E.D.

As a final remark, it is possible to notice that the proof above avoids the recourse to quasi-inverses and to the concept of equimeasurability ([1] §10.12) so that it simplifies the result of [5] beside generalizing it.

As a corollary one obtains (a part of) Theorem 3 of [5].

Corollary. The metric d_p in the space L^p with $p \in]1,+\infty[$ is given by

$$d_p(f,g) = \{ \int_{R_+} x^p \, dF_{fg}(x) \}^{1/p}$$

Proof. Choose $\varphi=\varphi_p$ to get, on account of (b) above,

$$d_p(f,g) := ||d(f,g)||_p = (p^{1/p}) \, ||d(f,g)||_{\varphi_p} =$$

$$= (p^{1/p}) \inf\{k>0 : p^{-1} \int_R (|x|/k)^p \, dF_{fg}(x) \leqslant 1\} =$$

$$= \{ \int_{R_+} x^p \, dF_{fg}(x) \}^{1/p} . \qquad \text{Q.E.D.}$$

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