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CONSTRUCTION OF 0-1 MATRICES ASSOCIATED TO PERIOD-DOUBLING PROCESSES

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ABSTRACT

We elaborate a method allowing the determination of 0-1 matrices corresponding to dynamics of the interval having stable, 2^k-periodic orbits, $n \in \mathbb{N}$. By recorrence on the finite dimensional matrices, we establish the form of the infinite matrices ($k \to \infty$).

0. Introduction.

Let us denote by $C[1]$, the set of all maps $f:[-1,1] \to [-1,1]$ such that:

1) f is of class C^1 in $[-1,1]$ and of class C^3 in $[-1,o[\cup]o,1]$.

2) f is even.

3) $f(-1)=-1$.

4) $f'(-1)>1$.

5) $f'(x) \neq 0$ for all $x \neq 0$.

6) $S(f)<0$ on $(-1,o) \cup (o,1)$ where we have denote by

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

the Schwartzian derivative of f.

7) For every $k \in N$, f possesses a periodic point of period 2^k.

8) f has no other period points which are not multiples of 2.

This paper will focus mainly on the construction of a infinite 0-1 matrix which is equivalent to the dynamics on the interval defined by the iterates of a quadratic map satisfying the conditions above. This we will do upon accumulation of 0-1 matrices corresponding to stable, 2^k-periodic dynamics.

1. Notation.

For any map $f \in C$, we will study the orbit of a critical point (c=o), $\{x_k\}_{k \in N}$ where $x_k = f^k(c) = f o \ldots o f(c)$. To this orbit, which completely characterizes the dynamic, we will associate a symbolic sequence

$$S=\{S_k\}_{k \in N_o}$$

where:

$$S_o = C$$

$$S_k = L \text{ if } f^k(c)<c$$

$$S_k = C \text{ if } f^k(c) = c$$

$$S_k = R \text{ if } f^k(c)>c$$

We introduce a order relation on the set of symbols (here R-parity) will mean parity of N_R, the number of times the iterates of c fall in the subinterval [c,1]):

$$L<c<R \quad \text{if} \quad N_R \text{ is even}$$

$$R<c<L \quad \text{if} \quad N_R \text{ is odd}$$

σ will stand for the shift-operation in the Σ-space, $\Sigma=\{L,C,R\}^{N_o}$

and $\sigma S = S' = \{S'_k\}_{k \in N_o}$ where $S'_k = S_{k+1}$.

We shall define the Kneading sequence [2] as the truncation:

$$S^{(k)} = S_1 S_2 \ldots S_{k-1} C \text{ of the shift } \sigma(CS_1 S_2 \ldots S_{k-1} CS_1 S_2 \ldots)$$

of the symbolic sequence determined by the itinerary of the critical point if the orbit has period K or, in case of a non-periodic orbit S, as the shift of the symbolic sequence $\sigma(S)$.

Given any Kneading sequence $S^{(k)} = S_1 S_2 \ldots S_k$ a $*$-product [3] is defined by:

$$S^{(k)} * R = S^{(k)} R S^{(k)} \text{ if } N_R \text{ is even}$$

$$S^{(k)} * R = S^{(k)} L S^{(k)} \text{ if } N_R \text{ is odd.}$$

2. Topological Markov Chain associated to a k-periodic orbit of the critical point.

In the following, we study a method of construction of a topological Markov chain [4] associated to any periodic orbit of the critical point.

Define:

$$X^{(k)} = \{x_i : x_i = f^i(c),\ i = o, \ldots, k-1 \text{ and } f^k(c) = c\}$$

$X^{(k)}$ is the orbit, of period k, of the critical point c. Associate to each x_i a symbolic sequence $\sigma^i(S) = \underbrace{\sigma o \ldots o \sigma}_{i \text{ times}}(S)$.

Also, denote the ordered sucession of the x_i by:

$Y^{(k)} = \{y_j = x_{i_j} = f^{i_j}(c) : x_{i_j} < x_{i_{j+1}},\ j \in \{o, \ldots, k-1\}$ and $i_j \in \{o, \ldots, k-1\}$, and likewise, associate to each y_j a symbolic sequence $\sigma^{i_j}(S)$. $Y^{(k)}$ determines a partition, $P^{(k)}$, of the interval:

$$I=[f^2(c),\ f^1(c)]=[y_o,y_{k-1}]=[x_2,x_1].$$

$$P^{(k)}=\{I_1=[y_o,y_1],\ I_2=[y_1,y_2],\dots,I_{k-1}=[y_{k-2},y_{k-1}]\}$$

and let $A=[a_{ij}]$ represent the transition matrix which f introduces, where

$$a_{ij}=\begin{cases}1 & \text{if } I_j\subset f(I_i)\\ o & \text{otherwise}\end{cases}$$

$\Sigma_{k-1}=\{I_1,I_2,\dots,I_{k-1}\}^{N_o}=\{1,2,\dots,k-1\}^{N_o}$ will be the space of symbolic sequences of "states" if $\omega\epsilon\Sigma_{k-1}$, where $\omega=\{\omega_n\}_{n\epsilon N_o}$, define the τ-shift operation

$$\tau:\Sigma_{k-1}\to\Sigma_{k-1} \text{ where } \tau\omega=\omega' \text{ with } \omega'_n=\omega_{n+1}$$

A symbolic dynamical system will be, by definition, the restriction (Σ_A,τ) of the τ-shift to a τ-invariant closed subset $\Sigma_A\epsilon\,\Sigma_{k-1}$, [4], [5].

Finally, if A is a (k-1)x(k-1) square matrix whose entries a_{ij} are zero or one only, define a topological Markov chain (Σ_A,τ) where:

$$\Sigma_A=\{\omega\epsilon\Sigma_{k-1},\ a_{\omega_n\omega_{n+1}}=1,\ \forall n\ \epsilon\ N_o\}$$

3. Construction of Markov Chains associated to period-doubling processes.

In this section we will apply some of the results of last section to study period-doubling processes [6] and build a succession of Markov chains associated to increasing 2^k-periodic orbits of the critical point.

It is known that, on a one-parameter family $f_a: I \to I$ of endomorphisms of the interval, monotone variation of the parameter a is followed by a sequence of values a_{2^k} for which the dynamics will only possess one super-stable periodic orbit ($f^{2^k}(c)=c$) of period 2^k, with $k=0,1,2,\ldots$, and such sequence will accumulate on a value a_{2^∞}. It must be noted that the "stability windows"—— neighbourhoods of values a_{2^k} where there exists a stable 2^k-periodic orbit —— will define the same Markov transition matrix: in fact, you just have to overlook the transient movement and consider only the stable assymptotic orbit !.

In terms of the $*$-product these orbits are [3]:

$$R;\ R*R;\ R*R*R;\ldots;\ R^{*^k};\ldots;\ R^{*^\infty}$$

corresponding to the kneading sequences $S^{(k)}$:

RC; RLRC; RLRRRLRC; RLRRRLRLRLRRRLRC;...

and the following symbolic sequences $\sigma^i(S^k)$ refering to these:

{RC,CR}; {RLRC, LRCR, RCRL, CRLR};...

{RLRRRLRC, LRRRLRCR, RRRLRCRL, RRLRCRLR, RLRCRLRR, LRCRLRRR, RCRLRRRL, CRLRRRLR};...

Also, as it was seen in §2, these correspond to the ordered sequences.

{CR,RC};{LRCR,CRLR,RCRL,RLRC};{LRRRLRCR,LRCRLRRR,
CRLRRRLR,RRLRCRLR,RRRLRCRL,RCRLRRRL,RCRCRLRR,RLRRRLRC};

These will in turn introduce the partitions $p^{(k)}$ and the transition matrices (fig. 1 & fig. 2).

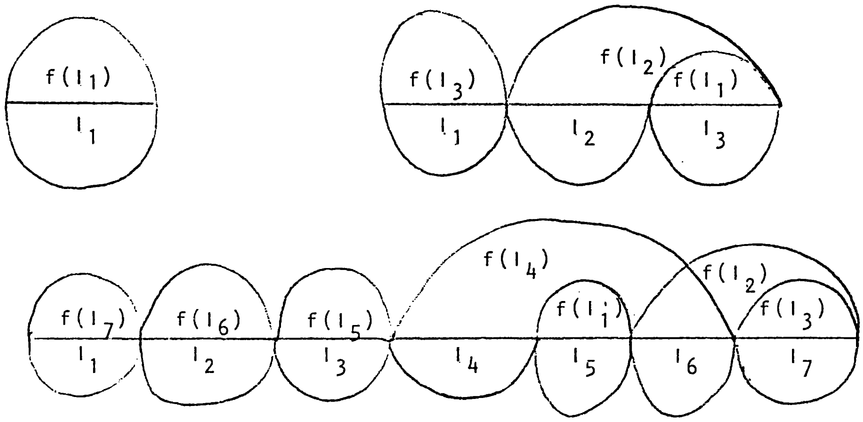

Fig. 1

$A_{2^1-1}=[1]$ $\quad A_{2^2-1}=\begin{bmatrix}0&0&1\\0&1&1\\1&0&0\end{bmatrix}$

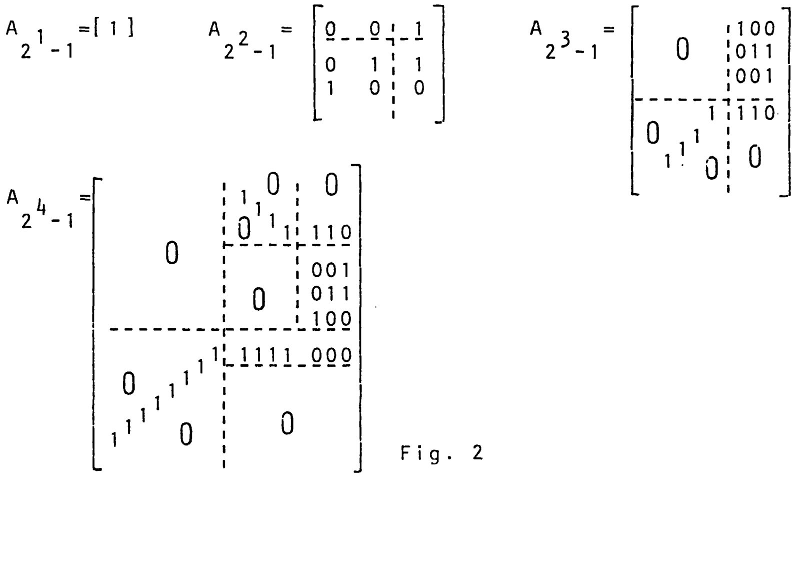

Fig. 2

In this way, for each k; a matrix A_{2^k-1} is defined:

$$A_{2^k-1} = \left[\begin{array}{c|cc} 0 & \hat{A}_{2^{k-1}-1} & \\ \hline B_{2^k-1} & C_{2^{k-2}} & 0 \\ & 0 & \end{array}\right] \quad \text{where } B_{2^k-1} = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}$$

$$C_{2^k-1} = [11\ldots1]$$

and $\hat{A}_{2^{k-1}-1}=[\hat{a}_{m\ell}]$ with $\hat{a}_{m\ell}=a_{ij}$, $\ell=j$, $m=2^{k-1}-i$

and $A_{2^{k-1}-1}=[a_{ij}]$.

To simplify the study of the accumulation ($k\to\infty$) of these matrices we shall redefine them by mere permutation of certain lines and columns. But let us first introduce the following notions:

A transitive "quasi-order" structure is introduced in the "state" space [5] by

$i<j \Leftrightarrow$ There exists a chain of state,

$i=i_o, i_1, \ldots, i_n=j$ such that $a_{i_o i_1 \ldots i_{n-1} i_n}>0$.

A state i will be called recurrent if $i<i$ and non-recurrent otherwise. Recurrent states are classified in equivalence classes:

$$i\sim j \Leftrightarrow i<j<i$$

Each such class defines a topological Markov sub-chain $(\Sigma_{A^{(R)}}, \sigma^{(R)})$ with:

$$\Sigma_{A^{(R)}}=\{\omega \in \Sigma^{(R)}; a_{\omega_n\omega_{n+1}}=1,\ \forall n \in N_o\};\ \sigma^{(R)}=\sigma|_{\Sigma^{(R)}}$$

$$\Sigma^{(R)}=\{i_1,\ldots,i_j\}^{N_o} \subset \Sigma_{k-1},\ j \leqslant k-1.$$

A equivalence class will be said final if there is no path that connects it to any other. A basic class is one for which the maximum eigenvalue $\lambda(A)$ (spectral radii) of A is equal to the one of $A^{(R)}$, $\lambda(A^{(R)})$, and a non-basic class will occur if $\lambda(A)>\lambda(A^{(R)})$, [9].

The T.M.C. (Σ_A,σ) and the corresponding A matrice will be said indecomposable or irreducible if all states are recurrent and form a final basic equivalence class.

The T.M.C. (Σ_A,σ) will be a primitive one if there is a n such that all entries in the matrix A^n are positive. It is known that a T.M.C. is topologicaly transitive (ergodic) if and only if it is indecomposable or irreducible and it is topologically mixing if and only if it is primitive.

Finally, one also, knows that a reducible non-negative matrix A has a permutation matrix P that allows one to reduce A to upper triangular form with block matrices $A^{(1)}$, $A^{(2)}$,...,$A^{(R)}$ [9] in the principal diagonal.

The transition matrices given as in fig. 2 will now appear as: (fig. 3).

$$\bar{A}_{2^1-1} = [\,1\,] = A^{(1)}$$

$$\bar{A}_{2^2-1} = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} A^{(1)} & W^{(2)} \\ \hline 0 & A^{(2)} \end{array}\right]$$

$$\bar{A}_{2^3-1} = \left[\begin{array}{c|cc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & 0 & & 0 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} \begin{array}{c|c} A^{(1)} & W^{(2)} \\ \hline 0 & A^{(2)} \end{array} & & W^{(3)} \\ \hline 0 & & A^{(3)} \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_{2^2-1} & W^{(3)} \\ \hline 0 & A^{(3)} \end{array}\right]$$

$$\bar{A}_{2^4-1} = \left[\begin{array}{c|c} \begin{array}{cc|c} \begin{array}{c|c} A^{(1)} & W^{(2)} \\ \hline 0 & A^{(2)} \end{array} & & W^{(3)} \\ \hline 0 & & A^{(3)} \end{array} & W^{(4)} \\ \hline 0 & \begin{array}{c|c} 0 & \begin{array}{c|c} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & 0 \\ \hline 0 & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \end{array} \\ \hline \begin{array}{ccc} 0 & & 1 \\ & 1 & 1 \\ 1 & & 0 \end{array} & 0 \end{array} \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_{2^3-1} & W^{(4)} \\ \hline 0 & A^{(4)} \end{array}\right]$$

FIG. 3

where $W^{(k)}$ represents a $(2^{k-1}-1)\times 2^{k-1}$ matrix that, under the dynamic perspective, represents a part of the transient movement.

In this way, for each k one will have a matrix $\bar{A}_{2^k-1}$ given by:

$$\bar{A}_{2^k-1}=\left[\begin{array}{c:c} \bar{A}_{2^{k-1}-1} & W^{(k)} \\ \hdashline 0 & A^{(k)} \end{array}\right]$$

$$A^{(k)} = \left[\begin{array}{c:c} 0 & \hat{A}^{(k-1)} \\ \hdashline B_{2^{k-2}} & 0 \end{array}\right]$$

With $\hat{A}^{(k-1)}=[\hat{a}_{m\ell}]$, $\hat{a}_{m\ell}=a_{ij}$, $\ell=j$, $m=2^{k-2}+1-i$ and $A^{(k-1)}=[a_{ij}]$. Also, the $W^{(k)}$ are of the form:

$$W^{(k)}=\left[\begin{array}{c|c|c|c|c|c} & U_1 & 0 & 0 & & 0 \\ & 0 & U_2 & & & \\ & & & Z_3 & & \\ 0 & & & U_3 & \ddots & \\ & & & & & Z_{k-1} \\ & & 0 & 0 & & U_{k-1} \\ & & & & & 0 \end{array}\right]$$

Where the $U_i=[11\ldots1]$, $1 \leqslant i \leqslant k-2$, are $1\times 2^{k-(i+2)}$ matrices and $U_{k-1}=[1]$. Also $Z_i=[z_{\ell(i),j(i)}]=[0]$, $3 \leqslant i \leqslant k-2$, are $2^{i-2}\times 2^{k-(i+2)}$ matrices and $Z_{k-1}=[0]$ is a $2^{k-3}\times 1$ matrix.

The following result stems from the above construction.

<u>Proposition.</u> The $\bar{A}_{2^\infty}$ matrix corresponds to the $R^{*\infty}$ kneading sequence (except for permutations on the intervals in $P^{(\infty)}$).

<u>Remark 1.</u> By inspection on the matrix $\bar{A}_{2^{k-1}}$, one realizes that it decomposes in Sub-matrices $(A^{(1)},\ldots A^{(k)}, W^{(2)}\ldots W^{(k)})$ where $A^{(i)}$ corresponds to the 2^{i-1}-periodic part and the $W^{(i)}$ correspond to the erratic part of the dynamics.

In the limit $k\to\infty$ one obtains a Markov subchain by considering only the final component

$$A^{(\infty)} = \lim_{k\to\infty} A^{(k)}$$

<u>Lemma.</u> If $f\in \mathcal{C}$, there exists a set $S \subset I$ such that $f|_S$ is topologically conjugate to

$$(\Sigma_{A^{(\infty)}}, \tau_{A^{(\infty)}}).$$

Proof. Misiurewicz defines a set S such that $S=\Omega\setminus Per(f)$ where Ω stands for the set of non-erratic points and $Per(f)$ is the set of periodic points of f.

We need to show that in the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ f|_S\ } & S \\ \pi\downarrow & & \downarrow\pi \\ \Sigma_{A^{(\infty)}} & \xrightarrow{\ \tau_{A^{(\infty)}}\ } & \Sigma_{A^{(\infty)}} \end{array}$$

π is a homeomorfism.

To each $X \in S$ corresponds one and only one $\omega \in \Sigma_{A^{(\infty)}}$ (evident). To show that to each $\omega \in \Sigma_{A^{(\infty)}}$ corresponds inversely one and only one $x \in S$ we start by noticing that a $x \in I$ such that $\pi(x)=\omega$ allways exists by definition of Σ_A. But such x belongs to S because otherwise it would have been erratic ($x \in W$) or periodic ($x \in Per(f)$) and in that case $W(x)$ wouldn't belong to $\Sigma_{A^{(\infty)}}$. Finally, if we suppose that there are two points $x,y \in S$ such that $x \neq y$ and $\omega(x)=\omega(y)$, then, owing to the fact that $\omega(x)=\{\omega_n(x)\}_{n\in N_o}$ and $\omega(y)=\{\omega_n(y)\}_{n \in N_o}$ and that S is a Cantor set, we should conclude that $w_o(x)$ and $w_o(y)$ define two distinct point set intervals $I_x=\{x\}$ and $I_y=\{y\}$, and that conclude the proof.

<u>Corollary.</u> The dynamic $(S,f|_S)$, $f \in C$ is ergodic but non-weakly mixing.

Proof. It is well known [1] that this dynamic is ergodic and has null topological entropy ($h_T=0$). On the other hand, we have constructed, via the lemma in last section, a infinite Markov process topologicaly equivalent to that dynamic.

From the stand point of ergodic theory, weakly mixing Markov chains are k-systems. [11]. As $h_T=\sup h_m$, where h_m=metric entropy, we conclude that $h_m=0$, $\forall m$. The result follows from the fact that the entropy of a k-system is strictly positive.

<u>Remark 2.</u> This result could also have been achieved through the existing isomorphism between $(S,f|_S)$ and the "adding Machine", using known results of ergodic theory [1], [13] or [12].

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