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ON A FUNCTIONAL EQUATION CONNECTED TO SUM FORM NONADDITIVE INFORMATION MEASURES ON AN OPEN DOMAIN-III*

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ABSTRACT

This is a part of a recent program by a number of authors to study information measures on open domain. In this series, this paper is devoted to the study of the functional equation (2) on an open domain. This functional equation is connected to the weighted entropy and weighted entropy of degree α.

§1. Introduction.

Let $\Gamma_n^0=\{P = (p_1,p_2,...,p_n)|0<p_k<1, \sum_{k=1}^{n} p_k = 1\}$ be the set of all finite complete discrete probability distributions. Let R be the set of real numbers and $R_+=\{x \in R|x > 0\}$. In analyzing the weighted additivity property of the weighted entropy [3,5.6] one comes across the following functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j,u_iv_j)=\sum_{j=1}^{m} q_jv_j\sum_{i=1}^{n} f(p_i,u_i)+\sum_{i=1}^{n} p_iu_i.\sum_{j=1}^{m} f(q_j,v_j) \quad (1)$$

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where $\sum_{i=1}^{n} p_i = 1 = \sum_{j=1}^{m} q_j, p_i, q_j \geqslant 0$ and $u_i, v_j \in R_+$. A generalization of (1) is the following:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(q_i q_j, u_i v_j) = \sum_{j=1}^{m} q_j v_j \sum_{i=1}^{n} g_i(p_i, u_i) + \sum_{i=1}^{n} p_i^{\alpha} u_i \sum_{j=1}^{m} h_j(q_j, v_j) \quad (2)$$

where $\alpha \in R \setminus \{0\}$. The weighted entropy of degree α (see [4]) with sum property and weighted -additivity gives rise to the functional equation (2) for $g_i = h_j = f$. For details refer [6].

In [6], a special case of the functional equation (2) was solved with the use of 0-probability and 1-probability when $g_i = h_j = f$ and f was measurable. The use of these extreme values of the probabilities makes the functional equation easily solvable. However, the use of these also requires the definitions like $0^{\beta} = 0$, $0 \log 0 = 0$. It is also a priori quite possible that there may exist solutions other than those on $[0,1]$ restricted to $]0,1[$ (see [2]). This is part of a recent program by a number of authors to study information measures on open domains (see [1,2,7]). In this paper, in line with results obtained on open domains of similar equations [7], we will find the measurable solution of (2) on an open domain.

§2. Some Auxiliary Results.

In order to prove our main theorem we need the following result.

<u>Result 1.</u> Let $f, g_i, h_j, \ell_j :]0,1[\rightarrow R (i=1,2,\ldots,n,\ j=1,2,\ldots,m)$ be measurable. They satisfy the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g_i(p_i) + \sum_{j=1}^{n} \ell_j(q_j) + \sum_{i=1}^{n} p_i^{\alpha} \cdot \sum_{j=1}^{m} h_j(q_j), \alpha \neq 1 \quad (3)$$

for a fixed pair of positive integers (m,n) $(\geqslant 3)$ and $P \in \Gamma_n^0, Q \in \Gamma_m^0$

if and only if,

$$\left.\begin{aligned} f(p) &= Ap\log p + B\log p + Cp^{\alpha} + (a+b-mnc)p+c,\\ g_i(p) &= Ap\log p + Bm\log p+Dp^{\alpha}+(a-\sum_{k=1}^{n} a_k)p+a_i,\\ \ell_j(p) &= Ap\log p + Bn\log p + (b-\sum_{r=1}^{m} b_r)p+b_j,\\ h_j(p) &= Cp^{\alpha}-(D+\sum_{s=1}^{m} c_s)p + c_j, \end{aligned}\right\} \quad (4)$$

where $A,B,C,D,a,b,c,a_i,b_j,c_j$ are arbitrary constants.

Proof: First we will treat the case $m=n$. Now interchanging $(p_1,p_2,\dots,p_m)$ and $(q_1,q_2,\dots,q_m)$ in (3) and using the symmetry of the left hand side of (3), we get

$$\begin{aligned} \sum_{i=1}^{m} g_i(p_i)+\sum_{j=1}^{m}\ell_j(q_j)+\sum_{i=1}^{m}p_i^{\alpha}\sum_{j=1}^{m}h_j(q_j) &= \sum_{j=1}^{m}g_j(q_j)+\sum_{i=1}^{m}\ell_i(p_i)\\ &+\sum_{j=1}^{m}q_j^{\alpha}\sum_{i=1}^{m}h_i(p_i). \end{aligned} \quad (5)$$

Letting $q_j = \frac{1}{m}$ for $j=1,2,\dots,m$ in (5), we obtain $\sum_{i=1}^{m} H_i(p_i)=0$, where

$$H_i(p) = g_i(p) - A_1p^{\alpha}-\ell_i(p)-m^{(1-\alpha)}h_i(p)-c_1p,\ p\in\,]0,1[\,.$$

Thus by Proposition 1 of [7], we get

$$g_i(p) = A_1p^{\alpha}+\ell_i(p) + m^{(1-\alpha)}h_i(p) + dp + d_i, \quad (6)$$

where A_1,c,d,d_i are constants. Putting (6) back into (5), we obtain

$$\begin{aligned} &(m^{1-\alpha}-\sum_{j=1}^{m}q_j^{\alpha})\cdot\sum_{i=1}^{m}h_i(p_i)-(m^{1-\alpha}-\sum_{i=1}^{m}p_i^{\alpha})\cdot\sum_{j=1}^{m}h_j(q_j)+\\ &A_1(\sum_{i=1}^{m}p_i^{\alpha}-\sum_{j=1}^{m}q_j^{\alpha}) = 0. \end{aligned} \quad (7)$$

Choose $(q_1^*, q_2^*, \ldots, q_m^*)$ in Γ_m^0 such that $m^{1-\alpha} - \sum_{j=1}^{m} q_j^{*\alpha} \neq 0$.

Letting $Q = Q^*$ in (7), we get (noting $\alpha \neq 1$), $\sum_{i=1}^{m} [h_i(p_i) - A_2 p_i^\alpha - B_2 p_i] = 0$. Again by Proposition 1 of [7], we get

$$h_i(p) = A_2 p^\alpha + B_2 p + ep + e_i, \tag{8}$$

where A_2, B_3, e, e_i are constants with $e + \sum_{i=1}^{m} e_i = 0$. Putting (8) into (6), we obtain

$$g_i(p) = A_3 p^\alpha + \ell_i(p) + B_3 p + t_i, \tag{9}$$

where A_3, B_3, and t_i $(i = 1,2,\ldots,m)$ are constants. Putting (8) and (9) into (3) and defining

$$F(p) := f(p) - A_2 p^\alpha - (B_3 + \sum_{r=1}^{m} t_r) p, \tag{10}$$

we get

$$\sum_{i=1}^{m} \sum_{j=1}^{m} F(p_i q_j) = (A_3 + B_2) \cdot \sum_{i=1}^{m} p_i^\alpha + \sum_{i=1}^{m} \ell_i(p_i) + \sum_{j=1}^{m} \ell_j(q_j) \tag{11}$$

Using the summetry of the left hand side of (11), we get $A_3 + B_2 = 0$. Then using Theorem 1 of [7], we get

$$\left.\begin{aligned} f(p) &= A_4 p \log p + A_5 p^\alpha + B_4 p + D_1 \log p + C_4, \\ \ell_i(p) &= A_4 p \log p + B_5 p + D_1 m \log p + D_i. \end{aligned}\right\} \tag{12}$$

Now, f, g_i, h_i, ℓ_i given by (12), (9), and (8) satisfy (3), provided they are of the form (4).

Now we consider the case <u>$m \neq n$.</u> Suppose $n > m$ and let $k = n - m$. Let us choose a $\beta \in \,]0,1[$ such that $k\beta < 1$. Setting

$p_{m+1}=p_{m+2}=\dots=p_n=\beta$, $1-k\beta=\delta$, (3) can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{m} f(p_iq_j)+k\sum_{j=1}^{m} f(\beta q_j) = \sum_{i=1}^{m} g_i(p_i)+\sum_{i=m+1}^{n} g_i(\beta) + \sum_{j=1}^{m}\ell_j(q_j)+k\beta^{\alpha}\sum_{j=1}^{m}h_j(q_j)+\sum_{i=1}^{m}p_i^{\alpha}\sum_{j=1}^{m}h_j(q_j) \tag{13}$$

for all $p_i,q_j>0$ with $\sum_{i=1}^{m} p_i = \delta$, $\sum_{j=1}^{m} q_j = 1$.

Letting $x_i = p_i/\delta (i=1,2,\dots,m)$ for $p_i \in]0,\delta[$ (13) becomes

$$\sum_{i=1}^{m}\sum_{j=1}^{m} f(\delta x_iq_j)+k\sum_{j=1}^{m} f(\beta q_j)=\sum_{i=1}^{m} g_i(\delta x_i)+\sum_{j=m+1}^{n} g_i(\beta) + \sum_{j=1}^{m}\ell_j(q_j)+k\beta^{\alpha}\sum_{j=1}^{m}h_j(q_j)+\delta^{\alpha}\sum_{j=1}^{m}x_i^{\alpha}\sum_{j=1}^{m}h_j(q_j) \tag{14}$$

with $q_j,x_i \in]0,1[$, $\sum_{i=1}^{m} x_i=1=\sum_{j=1}^{m} q_j$. Defining

$$\left.\begin{aligned} &F(x) := f(\delta x),\ G_i(x) := g_i(\delta x),\ H_j(x) := \delta^{\alpha}h_j(x)\\ &L_j(x) := k\beta^{\alpha}h_j(x)+\ell_j(x) + \sum_{i=m+1}^{n} g_i(\beta)-kf(\beta x)\end{aligned}\right\} \tag{15}$$

for all $x \in]0,1[$, (14) takes the form

$$\sum_{i=1}^{m}\sum_{j=1}^{m} F(x_iq_j)=\sum_{i=1}^{m} G_i(x_i)+\sum_{j=1}^{m} L_j(q_j)+\sum_{i=1}^{m} x_i^{\alpha}\sum_{j=1}^{m} H_j(q_j) \tag{16}$$

The measurable solution of (16) can be obtained from the previous case $m = n$ and then using (15), we have

$$\left.\begin{aligned} f(p) &= Ap\log p + B\log p + Cp^{\alpha} + Ep + c \\ g_i(p) &= Ap\log p + B_1\log p + Dp^{\alpha} + E_1p + a_i \\ \ell_j(p) &= Ap\log p + B_2\log p + E_2p + b_j \\ h_j(p) &= Cp^{\alpha} + E_3p + c_j \end{aligned}\right\} \qquad (17)$$

where $A,B,B_1,B_2,C,D,E,E_1,E_2,E_3,a,c,a_i,b_j,c_j$ are constants. Now f,g_i,ℓ_i,h_j given by (17) satisfy (3) provided they are of the form (4). The subcase $n<m$ can be argued similarly. The if part of the proof is easy to verify. This completes the proof of Result 1.

Note the dependence of the solution on m and n.

§3. The solution of the equation (2).

In this section, we shall prove the following theorem.

<u>Theorem 2.</u> Let f,g_i,h_j: $]0,1[\times R_+ \to R$ $(i = 1,2,\ldots,n, j=1,2,\ldots m)$ be measurable in each variable and satisfy the functional equation (2), where $P \in \Gamma_n^0$ and $Q \in \Gamma_m^0$, for some fixed pair m,n ($\geqslant 3$) and $\alpha \neq 0$. Then

$$\left.\begin{aligned} f(p,u) &= Aup^{\alpha} + kup - mncp + c \\ g_i(p,u) &= A'up^{\alpha} + kup - nbp + b_i \\ h_j(p,u) &= Aup^{\alpha} - A'up - mdp + c_j \end{aligned}\right\} \text{ for } \alpha \neq 1 \qquad (18)$$

and

$$\left.\begin{aligned} f(p,u) &= Aup\log p + Bpu\log u + (C+D)up - map + a \\ g_i(p,u) &= Aup\log p + Bpu\log u + Cup - nbp + b_i \\ h_j(p,u) &= Aup\log p + Bpu\log u + Dup - mdp + c_j \end{aligned}\right\} \text{for } \alpha=1 \quad (19)$$

where A,B,C,D, are arbitrary constants and b,d,b_i,c_j are constants satisfying $\sum_{j=1}^{n} b_i - nb = 0 = \sum_{j=1}^{m} c_j - md$.

<u>Proof.</u> Let us substitute $u_i = u$ and $v_j = v$ in (2) to obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{f(p_i q_j, uv)}{uv} = \sum_{i=1}^{n} \frac{g_i(p_i, u)}{u} + \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} \frac{h_j(q_j, v)}{v} \tag{20}$$

for all $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$ and $u, v \in R_+$.

First we will consider the case <u>$\alpha \neq 1$.</u> We temporarily fix u and v in (20). Then by Result 1, the measurable solution of (20) is given by

$$\left.\begin{aligned} f(p,u) &= A(u)up^{\alpha} + B(u)up + C(u)u, \\ g_i(p,u) &= A'(u)up^{\alpha} + B_i(u)up + b_i(u)u, \\ h_j(p,u) &= A(u)up^{\alpha} + c_j(u)up + d_j(u)u. \end{aligned}\right\} \tag{21}$$

Substitution of (21) into (20) gives

$$\begin{aligned} A(uv) \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\alpha} + B(uv) + mnC(uv) &= A'(u) . \sum_{i=1}^{n} p_i^{\alpha} + \sum_{i=1}^{n} B_i(u) p_i \\ &+ \sum_{i=1}^{n} b_i(u) + \sum_{i=1}^{n} p_i^{\alpha} [A(v) . \sum_{j=1}^{m} q_j^{\alpha} + \sum_{j=1}^{m} c_j(v) q_j \\ &+ \sum_{j=1}^{m} d_j(v)]. \end{aligned} \tag{22}$$

Comparing the coefficients of $\sum_{i=1}^{n} p_i^{\alpha}$ in (22), we get

$$A(uv) \sum_{j=1}^{m} q_j^{\alpha} = A'(u) + A(v) \sum_{j=1}^{m} q_j^{\alpha} + \sum_{j=1}^{m} c_j(v) q_j + \sum_{j=1}^{m} d_j(v) \tag{23}$$

and

$$B(uv) + mnC(uv) = \sum_{i=1}^{n} B_i(u) p_i + \sum_{i=1}^{n} b_i(u). \tag{24}$$

Similarly from (23), we have

$$A(uv) = A(v) \text{ (coefficient of } \sum_{j=1}^{m} q_j^{\alpha}) = A \text{ (say)}$$

and

$$A'(u) + \sum_{j=1}^{m} c_j(v)q_j + \sum_{j=1}^{m} d_j(v) = 0. \tag{25}$$

From (25), first with v=1 results $A'(u)$ = constant = A' (say) and then $c_1(v)(q_1-q_1^*) + c_2(v)(q_2-q_2^*) = 0$ (Choosing $Q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ in Γ_m^0 with $q_1+q_2=q_1^*+q_2^*$, $q_r=q_r^*$, $r=3,4,\ldots,m$ and substitute Q^* in (25) and subtract the resulting equation from (25)). That is, $c_1(v) = c_2(v)$ or

$$c_j(v) = D(v) \text{ for } j = 1,2,\ldots,m. \tag{26}$$

Similarly from (24), we get $B(v) + mnC(v)$ = constant = k (say) (letting u = 1 in (24)) and $\sum_{i=1}^{n} B_i(u)p_i + \sum_{i=1}^{n} b_i'(u) = k$, that is, $B_i(u) = E(u)$ for $i = 1,2,\ldots,n$ (Use similar argument as above). Thus, we have

$$\left.\begin{aligned} A(u) &= A,\ A'(u) = A',\ B_i(u) = E(u), \\ c_j(v) &= A(v),\ B(v) = k-mnC(v), \\ E(u) &= k-\sum_{s=1}^{n} b_s(u),\ D(v) = -A' - \sum_{t=1}^{m} d_t(v). \end{aligned}\right\} \tag{27}$$

Thus (21) can be rewritten as

$$\left.\begin{aligned} f(p,u) &= Aup^{\alpha} + (k-mnC(u))up + C(u)\,u, \\ g_i(p,u) &= A'up^{\alpha} + (k-\sum_{s=1}^{n} b_s(u))up + b_i(u)u, \\ h_j(p,u) &= Aup^{\alpha} - (A' + \sum_{t=1}^{m} d_t(u))up + d_j(u)u, \end{aligned}\right\} \tag{28}$$

where A, A', k are constants. Now by putting (28) into (2) for $\alpha \neq 1$, one obtains

$$\sum_{i=1}^{n}\sum_{j=1}^{m} C(u_i v_j)u_i v_j(1-mnp_i q_j)=(\sum_{j=1}^{m} v_j q_j).\sum_{i=1}^{n} u_i[b_i(u_i)- \tag{29}$$

$$- p_i \sum_{s=1}^{n} b_s(u_i) + (\sum_{i=1}^{n} u_i p_i^{\alpha})\sum_{j=1}^{m} v_j[d_j(v_j)-q_j \sum_{t=1}^{m} d_t(v_j)]$$

for all $u_i, v_j \in R_+$, $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$. Letting $u_i = 1$, $i=1,2,\ldots,n$ in (29) and then comparing the coefficients of Σp_i^{α}, we get

$$\sum_{j=1}^{m} d_j(v_j)v_j - \sum_{j=1}^{m}(\sum_{s=1}^{m} d_s(v_j))v_j q_j = 0. \tag{30}$$

We choose $Q^* = (q_1^*, q_2^*, \ldots, q_m^*)$ such that $q_1 + q_2 = q_1^* + q_2^*$, $q_r = q_r^*$, $r = 3,4,\ldots,m$. Letting $Q = Q^*$ into (30) and subtracting the resulting equation from (30), we get

$$\sum_{s=1}^{m} d_s(v_1)v_1 = \sum_{s=1}^{m} d_s(v_2)v_2.$$

That is $\sum_{s=1}^{m} d_s(v)v =$ constant. So (30) gives $\sum_{j=1}^{m} d_j(v_j)v_j = D$ i.e. $d_j(v)v =$ constant $= c_j$ (say). Similarly letting $u_i = 1$ and $d_j(v)v = c_j$ in (29), we get

$$\sum_{j=1}^{m} C(v_j)v_j = m \sum_{j=1}^{m} C(v_j)v_j q_j \tag{31}$$

(with $p_i = \frac{1}{n}$). As before, choosing Q^* etc. we get $C(v)v =$ constant $= C$ (say). This in (29) gives

$\sum_{i=1}^{n} b(u_i)u_i - \sum_{i=1}^{n}(\sum_{s=1}^{n} b_s(u_i))p_i = 0$, i.e. $b_i(u)u = b_i$. Thus from (28) we get (18) with $\sum_{i=1}^{n} b_i - nb = 0 = \sum_{j=1}^{m} c_j - md$.

Now we consider the case $\alpha = 1$. Then the equation (20) reduces to

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{f(p_iq_j,uv)}{uv}=\sum_{i=1}^{n}\frac{g_i(p_i,u)}{u}+\sum_{j=1}^{m}\frac{h_j(q_j,v)}{v}. \tag{32}$$

Keeping u and v temporarily fixed in R_+, the measurable solution of (32) can be obtained from [7] as

$$\left.\begin{aligned} f(p,u) &= A(u)up\log p + E(u)u\log p + F(u)up + G(u)u,\\ g_i(p,u) &= A(u)up\log p + mE(u)u\log p + M_i(u)up + N_i(u)u_i,\\ h_j(p,u) &= A(u)up\log p + nE(u)u\log p + S_j(u)up + T_j(u)u, \end{aligned}\right\}$$

where $A,E,F,G,M_i,N_i,S_jT_j:R_+\to R$ are real valued functions of u. Letting (33) into (32), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m}A(uv)p_iq_j\log(p_iq_j)+\sum_{i=1}^{n}\sum_{j=1}^{m}E(uv)\log(p_iq_j)+\sum_{i=1}^{n}\sum_{j=1}^{m}F(uv)p_iq_j$$

$$+mnG(uv)=\sum_{i=1}^{n}A(u)p_i\log p_i+m\sum_{i=1}^{n}E(u)\log p_i+\sum_{i=1}^{n}M_i(u)p_i+\sum_{i=1}^{n}N_i(u)$$

$$+\sum_{j=1}^{m}A(v)\,q_j\log q_j+n\sum_{j=1}^{m}E(v)\,\log q_j+\sum_{j=1}^{m}S_j(v)q_j+\sum_{j=1}^{m}T_j(v). \tag{34}$$

From (34) comparing the coefficients of $p_i\log p_i$, $q_j\log q_j$, $\log p_i$ and $\log q_j$, we obtain

$$\left.\begin{aligned} A(uv) &= A(u) = A(v) = A, \text{ constant}\\ E(uv) &= E(u) = E(v) = E, \text{ constant} \end{aligned}\right\} \tag{35}$$

and

$$F(uv) + mnG(uv)=\sum_{j=1}^{n}M_i(u)p_i+\sum_{i=1}^{n}N_i(u)+\sum_{j=1}^{m}S_j(v)q_j+\sum_{j=1}^{m}T_j(v). \tag{36}$$

Keeping all fixed except p_i in (36) and letting $P = P^* = (p_1^*,p_2^*,\ldots,p_n^*)$ with $p_1+p_2 = p_1^*+p_2^*$ and $p_r^* = p_r$ for $r \geqslant 3$

into (36) and subtracting the resulting from (36), we get $M_1(u)(p_1-p_1^*)+M_2(u)(p_s-p_s^*) = 0$ i.e. $M_2(u) = M_1(u)=M(u)$ (say) i.e.

$$M_i(u) = M(u), \quad (i = 1,2,\dots,n) \text{ (say)}. \tag{37}$$

Similarly,

$$S_j(u) = S(u), \quad (j = 1,2,\dots,m). \tag{38}$$

Letting (37) and (38) into (36), we get

$$F(uv) + mnG(uv)=(M(u)+\sum_{i=1}^{n} N_i(u))+(S(v)+\sum_{j=1}^{m} T_j(v)) \tag{39}$$

for all $u,v \in R_+$, which is a Pexider equation whose solution is given by

$$\left.\begin{aligned} F(u) &= B \log u + C + D - mnG(u), \\ M(u) &= B \log u + C - \sum_{s=1}^{n} N_s(u), \\ S(u) &= B \log u + D - \sum_{t=1}^{n} T_t(u). \end{aligned}\right\} \tag{40}$$

Using (35), (37), (38), (40) and (33) in (2) for $\alpha = 1$, we get

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} G(u_iv_j)u_iv_j(1-mnp_iq_j)+E\sum_{i=1}^{n} u_i\log p_i \sum_{j=1}^{m} v_j + \\ &+E\sum_{i=1}^{n} u_i\sum_{j=1}^{m} v_j \log q_j=(\sum_{j=1}^{m} v_jq_j)\sum_{i=1}^{n} u_i[N_i(u_i) - \\ &-\sum_{s=1}^{n} N_s(u_i)p_i] + (\sum_{i=1}^{n} u_ip_i)\sum_{j=1}^{m} v_j[T_j(v_j) - \sum_{t=1}^{m} T_t(v_j)q_j]+ \\ &+nE\sum_{i=1}^{n} u_i \log p_i\sum_{j=1}^{m} v_jq_j+mE\sum_{i=1}^{n} u_ip_i\sum_{j=1}^{m} v_j \log q_j. \end{aligned} \tag{41}$$

First, equating the coefficients of $\sum_{i=1}^{n} u_i \log p_i$, we get E=0. Hence (41) reduces to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} G(u_i v_j) u_i v_j (1-mnp_i q_j) = (\sum_{j=1}^{m} v_j q_j) \sum_{i=1}^{n} u_i [N_i(u_i) - \sum_{s=1}^{n} N_s(u_i) p_i] +$$

$$+ (\sum_{i=1}^{n} u_i p_i) \sum_{j=1}^{m} v_j [T_j(v_j) - \sum_{t=1}^{m} T_t(v_j) q_j]. \tag{42}$$

Letting $u_i = u$ in (42), we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{m} G(uv_j) uv_j (1-mnp_i q_j) = u \sum_{j=1}^{m} v_j [T_j(v_j) - (\sum_{t=1}^{m} T_t(v_j)) q_j]. \tag{43}$$

Now $q_j = 1/m$ in (43) yields

$$\sum_{j=1}^{m} T_j(v_j) v_j = \frac{1}{m} \cdot \sum_{j=1}^{m} (\sum_{t=1}^{m} T_t)(v_j) v_j. \tag{44}$$

Putting $v_j = v$, and the rest of v_j's = 1, we get

$$\begin{aligned} T_j(v) &= \frac{1}{m}(T_1+T_2+\ldots+T_m)(v)v + c_j \\ &= T(v)+c_j \text{ (say)}, \quad j = 1,2,\ldots,m. \end{aligned} \tag{45}$$

Similarly

$$N_i(u)u = N(u) + d_i, \quad i=1,2,\ldots,n. \tag{46}$$

Now (43) with u = 1 and (45) yield

$$\sum_{j=1}^{m} [nG(v_j)v_j - T(v_j)] = m \sum_{j=1}^{m} [nG(v_j)v_j - T(v_j)] q_j$$

Setting $v_1 = v$, $v_2 = \ldots = v_m = 1$ in the above, we have

$$nG(v)v - T(v) = \text{constant} = k \text{ (say)}$$

that is,

$$G(v)v = \frac{1}{n} T(v) + \frac{k}{n} . \tag{47}$$

Similarly

$$G(v)v = \frac{1}{m}N(v) + \frac{k'}{m} . \tag{48}$$

Using (43) and (47), we get

$$\sum_{j=1}^{m} T(uv_j)(1-mq_j) = u \cdot \sum_{j=1}^{m} T(v_j)(1-mq_j).$$

Putting $v_1 = v$, $v_2 = 1 = \ldots = v_m$ in the above, we obtain

$$T(uv) - T(u) = uT(v) - uT(1).$$

From the above equation it is easy to see that

$$T(u) = k_1 u + k_2. \tag{49}$$

From (47), (48), (49) and (42), we have

$$k_1 \left(\sum_{j=1}^{m} v_j - m \sum_{j=1}^{m} q_j v_j\right)\left(\frac{1}{n} \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} u_i p_i\right) = 0.$$

From this we can conclude $k_1 = 0$ (for example choose p_i, q_j such that $\sum_{i=1}^{n} p_i^2 \neq \frac{1}{n}$, $\sum_{j=1}^{m} q_j^2 \neq \frac{1}{m}$, $v_j = q_j$ and $u_i = p_i$). Then $T(u) = k_2$, $G(v)v = \frac{k}{n} = k_3$, $N(v) = \frac{m}{n} \cdot k - k' = k_4$. The equations (33), (35), (37), (38), (40), (45), (46), (47), (48), (49) with E=0 in (2) yield (19).

This completes the proof of Theorem 2.

Corollary. Let $f:]0,1[\times R_+ \to R$ be measurable in each variable and satisfy the functional equation (2) for $f = g_i = h_j$ ($i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$), where $P \in \Gamma_n^0$ and $Q \in \Gamma_m^0$, for some fixed pair $m,n (\geqslant 3)$. Then f is given by

$$f(p) = \begin{cases} Au(p^{\alpha}-p) & , \alpha \neq 1 \\ Aup \log p + Bpu \log u, & \alpha = 1 \end{cases}$$

where A,B are arbitrary constants.

Remark. Note that the solutions of (2) depend upon m,n. However, if the functions are same i.e. $f = g_i = h_j$ then the solutions of (2) do not depend upon m,n which is obvious from Corollary.

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