STOCHASTICA Vol. IX Nº 2, 1985.

SOME PROPERTIES OF THE JACOBIAN sn z FUNCTION

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ABSTRACT

Using some results of the theory of functional equations we deduce some properties of the Jacobian sn z function which seems to be new. Also some functional equations had been found which are fulfilled by the sn z function which the author did not found in the literature.

It is known that the Jacobian ellyptic sn z function satisfies some interesting functional equations (see e.g. [1] p. 389 and p. 415). In the present paper we add to them some others which seems to be new and found some unknown properties of the Jacobian function.

Let k $(0 < k^2 < 1)$ be the parameter of the Jacobian ellyptic sn z function and denote as usually in the literature, by 4K and 2iK' the primitive periods. Its only singularities are at the points $2mK + (2n+1)K'i$ $(m,n = 0,\pm 1,\pm 2,\dots)$; this are simple poles with the residues $1/k$ and $-1/k$ respectively.

Consider the following function

$$Q(z) = \int_0^z \mathrm{sn}^2 t \, dt$$

where $z \neq 2mK + (2n+1)K'i$ $(m,n = 0,\pm1,\pm2,\ldots)$ otherwise arbitrary and the pass of integration is an arbitrary rectificable curve in the complex plane joining the origin to z deviating all the poles of sn z. By a well known theorem ([1]) p. 404) we see immediately that $Q(z)$ fulfills the following nonhomogeneous Cauchy functional equation

$$f(z_1+z_2) - f(z_1) - f(z_2) = \operatorname{sn} z_1 \operatorname{sn} z_2 \operatorname{sn}(z_1+z_2) \tag{1}$$

for all z_1, z_2 and z_1+z_2 which are not in the set of poles. As αz is the general analytic solution of the Cauchy functional equation

$$g(z_1+z_2) - g(z_1) - g(z_2) = 0$$

where α is an arbitrary constant, therefore

$$Q(z) + \alpha z$$

is the general analytic solution of (1).

1. Let now consider first the case in which z_1 and z_2 are reals x_1 and x_2. In this case $|\operatorname{sn} x| \leqslant 1$ $(x \in R)$ and of behalf of (1)

$$|Q(x_1+x_2) - Q(x_1) - Q(x_2)| \leqslant 1 \quad (x_1, x_2 \in R).$$

By a theorem of D. H. Hyers [2] it exists one and only one (real) constant α for which

$$|Q(x) - \alpha x| \leqslant 1 \qquad (x \in R).$$

This means

<u>Proposition 1.</u> The curve of $Q(x) = \int_0^x \operatorname{sn}^2 t \, dt$ remains in the

strip

$$\{\alpha x-1,\alpha x+1\}.$$

As by the theorem of Hyers α is uniquelly defined, on the other hand the Jacobian sn x function is uniquelly defined by k, this means that to every value $k \in (0,1)$ corresponds exactly one α, i.e. α is a function of k. It is easy to define this α. By a theorem di Hyers [2]

$$\alpha = \lim_{n\to\infty} (1/2^n x) \int_0^{2^n x} \mathrm{sn}^2 t \, dt.$$

This limit is independent of x. Obviously $0 < \alpha < 1$. $\mathrm{sn}^2 t$ is periodic with the period 2K. Let us put $x = K$ and by the periodicity we get

$$(2) \qquad \alpha = \lim_{n\to\infty} (1/2^{n-1}(2K)) \int_0^{2^{n-1}(2K)} \mathrm{sn}^2 t \, dt = (1/K) \int_0^K \mathrm{sn}^2 t \, dt.$$

But by a well known theorem ([1] p. 402) we conclude

$$\int_0^K \mathrm{sn}^2 t \, dt = (K/k^2) - (1/k^2) \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2} \, dt$$

therefore

$$(3) \qquad \alpha = (1/k^2) - (1/k^2 K) \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{1/2} \, dt.$$

In this way we have expressed α by the comlete ellyptic integral of the second kind.

On the other side it is known ([1] p. 393) that

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2} \, dt,$$

substituting this into (3), we get explicitly the expression of $\alpha=\alpha(k^2)$:

$$(4) \qquad \alpha = \alpha(k^2) = \frac{1}{k^2} - \frac{1}{k^2}\,\frac{\int_0^{\pi/2}(1-k^2\sin^2 t)^{1/2}\,dt}{\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}\,dt} .$$

Proposition 2. The number α defined in Proposition 1 is a function of k^2, its explicite form is given in (4) by means of complete ellyptic integrals of the first and second kind.

2. It is also interesting to see the limits of $\alpha(k^2)$ as $k \to 0$ resp. $k \to 1$. This limits will be used later on.

It is known ([1] p. 385) that if $k \to 0$, then $K \to \pi/4$ and $\mathrm{sn}\ t \to \sin t$ uniformly in an enough little neigborholld of the origin. On behalf of this we conclude by (2) that

$$(5) \qquad \alpha(0) = \lim_{k\to o} \alpha(k^2) = (2/\pi)\int_0^{\pi/2} \sin^2 t\ dt = 1/2.$$

Let us now see what happens if $k \to 1$. In order to calculate this limit we rewrite the expression (4) in the following way:

$$\alpha(k^2) = \frac{\int_0^{\pi/2} \sin^2 t\ (1-k^2\sin^2 t)^{-1/2}\,dt}{\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}\,dt}$$

Let us now consider an arbitrary number $\delta: 0 < \delta < 1$ and write

$$\alpha(k^2) = \frac{\int_0^{\delta} \sin^2 t\ (1-k^2\ \sin^2 t)^{-1/2}\,dt}{\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}\,dt} + $$

$$+ \frac{\int_\delta^{\pi/2} \sin^2 t\ (1-k^2\sin^2 t)^{-1/2}\,dt}{\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}\,dt} .$$

Obviously

$$1 \geqslant \frac{\int_{\delta}^{\pi/2} \sin^2 t\ (1-k^2 \sin^2 t)^{-1/2}\, dt}{\int_{0}^{\pi/2} (1-k^2 \sin^2 t)^{-1/2}\, dt} \geqslant$$

$$\sin^2\delta\ \frac{\int_{\delta}^{\pi/2} (1-k^2\sin^2 t)^{-1/2}\, dt}{\delta(1-k^2\sin^2\delta)^{-1/2} + \int_{\delta}^{\pi/2} (1-k^2\sin^2 t)^{-1/2}\, dt} =$$

$$\sin^2\delta\ \frac{1}{\frac{\delta}{K_\delta}(1-k^2\sin^2\delta)^{-1/2} + 1} \geqslant \frac{\sin^2\delta}{\frac{\delta}{K_\delta \cos\delta} + 1}$$

where

$$K_\delta = \int_{\delta}^{\pi/2} (1-k^2\sin^2 t)^{-1/2}\, dt.$$

Let us fix the value of δ and consider

$$K_\delta = \int_{\delta}^{\pi/2} (1-k^2\sin^2 t)^{-1/2}\, dt \geqslant \frac{(\pi/2) - \delta}{(1-k^2)^{1/2}} .$$

We see at once from this that $K_\delta \to \infty$ for $k \to 1$. Let us now chose k near enough to 1 in order to get

$$0 < \frac{\delta}{K_\delta \cos\delta} < 1,$$

then

$$\frac{1}{1 + \frac{\delta}{K_\delta \cos\delta}} \geqslant 1 - \frac{\delta}{K_\delta \cos\delta} .$$

Because this

$$1 \geqslant \alpha(k^2) \geqslant \sin^2\delta - \frac{\delta \sin^2\delta}{K_\delta \cos\delta} .$$

This inequality shows that for $k \to 1$

$$1 \geq \lim_{k \to \infty} \alpha(k^2) \geq \sin^2 \delta.$$

But δ is a number arbitrary close to $\pi/2$, therefore

$$\lim_{k \to 1} \alpha(k^2) = 1 \tag{7}$$

holds.

3. Let us now return to the functional equation (1) and let consider again real values for z_1 and z_2. As $Q(x)$ is a solution of (1), i.e. the solvability if (1) is obvious, we can apply a theorem of Jessen-Karpf-Thorup [4] which gives a necessary and sufficient condition in order to garantee the solvability of a functional equation of the type (1) in the case that the right hand side is a symmetric function. In our case this last condition is fulfilled. By applying the theorem quoted the following relation holds:

$$\begin{aligned} &\mathrm{sn}(z_1+z_2+z_3)[\,\mathrm{sn}\, z_2 \;\mathrm{sn}(z_1+z_3) - \mathrm{sn}\, z_1 \;\mathrm{sn}(z_2+z_3)\,] = \\ &\mathrm{sn}\, z_3[\,\mathrm{sn}\, z_2 \;\mathrm{sn}(z_2+z_3) - \mathrm{sn}\, z_1 \;\mathrm{sn}(z_1+z_3)\,] \qquad (z_1,z_2,z_3 \in R). \end{aligned} \tag{8}$$

if we introduce the new variables u,v,w as follows

$$u - v = z_1;\; u + v - w = z_2;\; w - u = z_3$$

then (8) takes the following form

$$\begin{aligned} &\mathrm{sn}\, u \;\mathrm{sn}(u+v-w)\;\mathrm{sn}(w-v) - \mathrm{sn}\, u \;\mathrm{sn}\, v \;\mathrm{sn}(u-v) = \\ &\mathrm{sn}\, v \;\mathrm{sn}(w-u)\;\mathrm{sn}(u+v-w) - \mathrm{sn}(w-u)\;\mathrm{sn}(u-v)\,\mathrm{sn}(w-v) \quad (u,v,w \in R). \end{aligned} \tag{9}$$

As sn z is an analytic function the relations (8) and (9) keeps their validity by the princip of analytic continuation also for all complex variables supposed that $z_1, z_2, z_3, z_1+z_3,\ z_2+z_3, z_1+z_2+z_3$ resp. $u, v, w, u-v, v-w, u+v-w$ are not poles of sn z.

Proposition 3. The Jacobian sn z function fulfills the functional equation (8) resp. (9).

They are other well known functional equations for sn z. E. g. ([1] p. 415).

(10) $\mathrm{sn}(v+w)\ \mathrm{sn}(v-w) + \mathrm{sn}(w+u)\ \mathrm{sen}(w-u) + \mathrm{sn}(u+v)\ \mathrm{sn}(u-v) +$

$$k^2\ \mathrm{sn}(v+w)\ \mathrm{sn}(v-w)\ \mathrm{sn}(w+u)\ \mathrm{sn}(w-u)\ \mathrm{sn}(u+v)\,\mathrm{sn}(u-v) = 0$$

or an other ([1] p. 389):

$$\mathrm{sn}(u+v)\ \mathrm{sn}(u-v)\ [1 - k^2\ \mathrm{sn}^2 u\ \mathrm{sn}^2 v] = \mathrm{sn}^2 u - \mathrm{sn}^2 v. \tag{11}$$

If we compare (10) and (11) with (8) (resp. (9)) we see that (10) and (11) containes explicitely the parameter k while (8) (and so (9)) not. This means (10) and (11) are fulfilled only by those sn z function which parameter occur in the functional equation, while (8) (resp. (9)) do not contain k, which means that (8) is fulfilled by all Jacobian sn z function independently of its parameter. Very probably (8) can be satisfied also by other analytic functions not only by the sn z function, but can also (10) and (11) satisfied by other analytic function?. For the time being this problems seems to be open.

4. Let us now apply the explicite expression of the solution of the functional equation (1) which is given in the paper [3]. In this way we get the following integrofunctional equation for sn x :

$$\int_0^x \mathrm{sn}^2 t\ dt = \alpha(k^2)x - \sum_{n=0}^{\infty} 2^{-n-1}(\mathrm{sn}\ 2^n x)^2\ \mathrm{sn}\ 2^{n+1}x\ (x \in R). \tag{12}$$

If $k \to 0$ we get by (5)

$$\int_0^x \sin^2 t\, dt = x/2 - \sum_{n=o}^{\infty} 2^{-n-1} \sin^2 2^n x \sin 2^{n+1} x, \tag{13}$$

or equivalently

$$\sin 2x = \sum_{n=o}^{\infty} 2^{-n+1} \sin^2 2^n x \sin 2^{n+1} x \quad (x \in R). \tag{14}$$

In order to get this last relation we have taken in (12) the limit term by term for $k \to 0$. This step is legal as the series (14) is uniformly convergent.

For $k \to 1$ as it is known $\mathrm{sn}\, x \to \mathrm{th}\, x$, this yields by (7) and (12)

$$\int_0^x \mathrm{th}^2 t\, dt = x - \sum_{n=o}^{\infty} 2^{-n-1} \mathrm{th}^2 2^n x\ \mathrm{th}\ 2^{n+1} x, \tag{15}$$

or equivalently

$$\mathrm{th}\ x = \sum_{n=o}^{\infty} 2^{-n-1} \mathrm{th}^2 2^n x\ \mathrm{th}\ 2^{n+1} x \quad (x \in R). \tag{16}$$

The formula (12) is also valid for all complex valued z for which $z \neq \frac{p}{2^{n-1}} K + \frac{2q+1}{2^n} K'i$ $(p,q = 0,\pm 1,\pm 2,\ldots;\ n=0,1,2,\ldots)$. This follows by the principe of analytic continuation. By this reason obviously (14) holds for every complex number z and (16) for all complex z different of $\frac{r\pi i}{2^{n+1}}$ $(r = \pm 1,\pm 2,\ldots;\ n=0,1,2,\ldots)$.

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Manuscript received in
November 30, 1983.

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