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HOW THE MAXIMUM OF GAUSSIAN RANDOM WALKS AND FIELDS IS INFLUENCED BY CHANGES OF THE VARIANCES

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ABSTRACT

In this paper we present an analytical proof of the fact that the maximum of gaussian random walks exceeds an arbitrary level β with a probability which is an increasing function of the step variances. An analogous result for stochastic integrals is also obtained.

KEY WORDS : Gaussian random walks, random fields, stochastic integrals.

1. Introduction.

In this note we consider a gaussian random walk:

$$S_r = \sum_{j=1}^{r} c_j X_j \qquad r = 1,2,\dots,n \tag{1}$$

where the X_j's are independent, standard, normal r.v.'s and the c_j's are non-negative constants.

Our aim here is to prove that if

$$\hat{c}_j > c_j \qquad \text{for any } j \tag{2}$$

and

$$\hat{S}_r = \sum_{j=1}^{r} \hat{c}_j X_j \tag{3}$$

then this inequality holds:

$$\text{Prob}\{\max_r S_r > \beta\} \leqslant \text{Prob}\{\max_r \hat{S}_r > \beta\} \tag{4}$$

The question arises because:

$$(\max_r S_r > \beta) \not\subset (\max_r \hat{S}_r > \beta)$$

when the $\hat{c}_j$'s are not proportional to the c_j's.

The reader can easily construct examples where changing the weights of a sample path overpassing β gives a new trajectory which does not exceed level β.

This is due to the fact that the new weights can increase the relative importance of negative steps thus letting the total displacement decrease below level β.

Although result (4) is an intuitively expected one, its proof is by no means a trivial matter. We must also observe that resorting to Slepian's lemma (see Marcus and Shepp (1970) or Marcus and Shepp (1971)) is here fruitless since this classical result concerns the comparison of gaussian processes and fields possessing equal variance.

This is clearly not the case and the question requires a completly different approach.

The random walk (1) can equivalently be written:

$$S_r = \sum_{j=1}^{r} Y_j \qquad r = 1,2,\ldots,n$$

where the Y_j's are zero-mean gaussian r.v.'s with variances c_j^2's. Result (4) can therefore be interpreted in another way, namely that the larger the variances of the steps Y_j, the larger the probability that the random walk exceeds an arbitrary level β.

2. The main result

For the random walks (1) and (3) the following inequality holds:

$$\mathrm{Prob}\{\max_r S_r > \beta\} \leqslant \mathrm{Prob}\{\max_r \hat{S}_r > \beta\}$$

Proof. Let us first consider this probability:

$$R = \mathrm{Prob}\{\max_r S_r > \beta\} =$$
$$= 1 - \int_A (2\pi)^{-\frac{n}{2}} (c_1 \dots c_n)^{-1} \exp\left\{- \sum_{i=1}^{n} \frac{x_i^2}{2c_i^2}\right\} dx_1 \dots dx_n$$

where:

$$A = \{x \in R^n : x_1 < \beta, x_1 + x_2 < \beta, \dots, x_1 + \dots + x_n < \beta\}$$

We must prove that the probability R is an increasing function of the arguments $c_1, c_2, \dots, c_n$.

Therefore we must analyse the following derivative:

$$\frac{\partial R}{\partial c_j} = \int_A (2\pi)^{-\frac{n}{2}} c_1^{-1} \dots c_{j-1}^{-1} \left(\frac{1}{c_j^2} - \frac{x_j^2}{c_j^4}\right) c_{j+1}^{-1} \dots c_n^{-1} \exp\left\{- \sum_{i=1}^{n} \frac{x_i^2}{2c_i^2}\right\} dx_1 \dots dx_n$$

Setting:

$$\begin{cases} x_1 = u_1 \\ x_1+x_2 = u_2 \\ \dots\dots\dots\dots \\ x_1+\dots+x_n = u_n \end{cases}$$

the above derivative can be converted into the following form:

$$\frac{\partial R}{\partial c_j} = \int_{-\infty}^{\beta} du_1 \int_{-\infty}^{\beta} du_2 \dots \int_{-\infty}^{\beta} du_n \; (2\pi)^{-\frac{n}{2}} \; c_1^{-1} \dots c_{j-1}^{-1} c_j^{-2} \left(1 - \frac{(u_j - u_{j-1})^2}{c_j^2}\right) \dots$$

$$\dots c_{j+1}^{-1} \dots c_n^{-1} \; \exp\left\{ - \sum_{i=1}^{n} \frac{(u_i - u_{i-1})^2}{2c_i^2} \right\} \qquad (7)$$

We now focus our attention on the inner part of the integral, namely on:

$$I_{j,n} = \int_{-\infty}^{\beta} du_{j+1} \cdot \int_{-\infty}^{\beta} du_n \; (2\pi)^{-\frac{(n-j)}{2}} \; c_{j+1}^{-1} \dots c_n^{-1} \; \exp\left\{ - \sum_{i=j+1}^{n} \frac{(u_i - u_{i-1})^2}{2c_i^2} \right\} (8)$$

Clearly $I_{j,n}$ is a function of u_j.

The integral with respect to u_n can be treated as follows:

$$\int_{-\infty}^{\beta} du_n \frac{1}{\sqrt{2\pi} c_n} \exp\left\{ - \frac{(u_n - u_{n-1})^2}{2c_n^2} \right\} = \qquad (9)$$

$$= \left(\frac{u_n - u_{n-1}}{c_n} = w_n \right)$$

$$= \int_{-\infty}^{\frac{\beta - u_{n-1}}{c_n}} \frac{1}{\sqrt{2\pi}} \exp\{ - \frac{w_n^2}{2} \} \; dw_n$$

The quantity (9) is a function of u_{n-1} which decreases from the value 1 to $\frac{1}{2}$ as the argument runs over $(-\infty, \beta)$.

Therefore the integral (9) is greater than or equal to $\frac{1}{2}$.

Repeating this argument we clearly obtain that:

$$I_{j,n} \geqslant \frac{1}{2^{n-j}} \quad \text{for any value of } u_j \text{ in } (-\infty, \beta) \tag{10}$$

The meaning of inequality (10) is straightforward. In fact it means that there is a positive probability that the random walk does not overpass the threshold β in n-j steps, provided that it starts from $u_j < \beta$.

We must now evaluate the integral with respect to the j-th variable, i.e. we must determine:

$$\int_{-\infty}^{\beta} c_j^{-2} \frac{1}{\sqrt{2\pi}} \left(1 - \frac{(u_j - u_{j-1})^2}{c_j^2}\right) \exp\left| - \frac{(u_j - u_{j-1})^2}{2c_j^2} \right| du_j = \tag{11}$$

$$= \left(\frac{u_j - u_{j-1}}{c_j} = w_j\right) = c_j^{-1} \int_{-\infty}^{\frac{\beta - u_{j-1}}{c_j}} (1 - w_j^2) \frac{1}{\sqrt{2\pi}} \exp\{-\frac{w_j^2}{2}\} dw_j =$$

$$= c_j^{-1} \left[\int_{-\infty}^{\frac{\beta - u_{j-1}}{c_j}} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{w_j^2}{2}\} dw_j + w_j \frac{1}{\sqrt{2\pi}} \exp\{-\frac{w_j^2}{2}\} \Big|_{-\infty}^{\frac{\beta - u_{j-1}}{c_j}} - \right.$$

$$\left. - \int_{-\infty}^{\frac{\beta - u_{j-1}}{c_j}} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{w_j^2}{2}\} dw_j \right] =$$

$$= \frac{1}{\sqrt{2\pi}\, c_j} \left(\frac{\beta - u_{j-1}}{c_j}\right) \exp\left\{ - \frac{(\beta - u_{j-1})^2}{2c_j} \right\}$$

When j=1, combining (10) and (11) we clearly obtain that:

$$\frac{\partial R}{\partial c_1} > 0$$

As far as the general case in concerned a further integration is necessary. Carrying out an integration w.r.t. u_{j-1} we obtain:

$$\frac{1}{2\pi}\frac{1}{c_j^2}\int_{-\infty}^{\beta}(\beta-u_{j-1})\exp\left\{-\frac{(\beta-u_{j-1})^2}{2c_j^2}-\frac{(u_{j-1}-u_{j-2})^2}{2c_{j-1}^2}\right\}du_{j-1} = \qquad (12)$$

$$= \quad (\beta-u_{j-1} = w) =$$

$$= \frac{1}{2\pi}\frac{1}{c_j^2}\int_{o}^{\infty} w\ \exp\left\{-\frac{w^2}{2c_j^2}-\frac{(\beta-w-u_{j-2})^2}{2c_{j-1}^2}\right\}dw > 0.$$

This suffices to prove that $\frac{\partial R}{\partial c_j} > 0$, since integrating (12) with respect to the remaining variables a positive quantity is clearly obtained.

With this at hand it is now a simple matter to conclude the proof of the theorem.

If:

$$S'_r = \hat{c}_1X_1+c_2X_2+\ldots+c_rX_r \qquad r=1,\ldots,n$$

the previous analysis authorizes us to write:

$$\text{Prob}\{\max_r S_r > \beta\} \leqslant \text{Prob}\{\max_r S'_r > \beta\}$$

Analogously if:

$$S''_r = \hat{c}_1X_1+\hat{c}_2X_2+c_3X_3+\ldots+c_rX_r$$

we have:

$$\text{Prob}\{\max_r S'_r > \beta\} \leqslant \text{Prob}\{\max_r S''_r > \beta\}$$

Repeating successively this argument we prove inequality (4).

<u>Corollary.</u> If $\{X_{h,k}, h=1,\dots,n;\ k=1,\dots,n\}$ are independent, zero-mean, normal variates, $c_{h,k} > 0$ are positive constants and:

$$S_{r,s} = \sum_{h=1}^{r} \sum_{k=1}^{s} c_{h,k} X_{h,k}$$

then, if

$$\hat{c}_{h,k} > c_{h,k} \qquad \text{for any } (h,k)$$

and

$$\hat{S}_{r,s} = \sum_{h=1}^{r} \sum_{k=1}^{s} \hat{c}_{h,k} X_{h,k}$$

the following is true:

$$\text{Prob}\{\max_{r,s} S_{r,s} > \beta\} \leqslant \text{Prob}\{\max_{r,s} \hat{S}_{r,s} > \beta\} \tag{13}$$

Proof. Ascertaining that (13) holds implies minor adjustments with respect to the proof of theorem 1.

3. The continuous counterpart.

Clearly the continuous counterpart of (1) is the stochastic integral:

$$Y(t) = \int_0^t g(s)\, dW(s) \tag{14}$$

where g is a real-valued, bounded, deterministic function.

In this case it is possible to obtain a result similar to that presented in theorem 1 with little effort.

Theorem 2. If $\hat{g}(s) \geq g(s)$ for $s \in [0,t]$, then the following result holds:

$$\text{Prob}\{\max_{0\leq z\leq t} \int_o^z g(s)dW(s) > \beta\} \leq$$

$$\leq \text{Prob}\{\max_{0\leq z\leq t} \int_o^z \hat{g}(s)dW(s) > \beta\} \qquad (15)$$

Proof. This can be shown in many ways. The simplest one is perhaps to point out that (14) is equivalent in distribution to a rescaled brownian motion.

We write this as follows:

$$Y(t) = B(\int_o^t g^2(s)ds)$$

B(.) being the standard brownian motion.

Therefore:

$$\text{Prob}\{\max_z \int_o^z g(s)dW(s) > \beta\} = \qquad (16)$$

$$= \text{Prob}\{\max_z B(\int_o^z g^2(s)ds) > \beta\} =$$

$$= 2\text{Prob}\{B(\int_o^t g^2(s)ds) > \beta\} \leq 2\text{Prob}\{B(\int_o^t \hat{g}^2(s)ds) > \beta\}$$

$$= \text{Prob}\{\max_z B(\int_o^t \hat{g}^2(s)ds) > \beta\} =$$

$$= \text{Prob}\{\max_z \int_o^z \hat{g}(s)dW(s) > \beta\}$$

<u>Remark.</u> The reason that the proof of (15) is quite straightforward while that of (4) requires a considerable amount of calculations is that stochastic integrals are path continuous processes displaying the Markov property.

The reflection principle then quickly implies the claimed result. Its discrete-time version, namely the random walk (1), does not permit us to employ the reflection principle because of jumps in the sample paths.

We finally observe that inequality (15) holds under a weaker assumption, that is if $\hat{g}(s)$ and $g(s)$ are such that:

$$\int_o^t \hat{g}^2(s)ds > \int_o^t g^2(s)ds$$

then (15) continues to be valid.

This appears clearly in the central steps of (16). Thus in the continuous case inequality (15) can hold also when the weight function $g(s)$ is not throughout dominated by $\hat{g}(s)$.

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