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ON SYMMETRIES AND PARALLELOGRAM SPACES

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ABSTRACT

The notion of a TST-space is introduced and its connection with a parallelogram space is given. The existence of a TST-space is equivalent to the existence of a parallelogram space, which is a new characterization of a parallelogram space. The structure of a TST-space is described in terms of an abelian group.

In their common paper [2], F. Ostermann and J. Schmidt introduced the notion of a parallelogram space and among other topics investigated its relationship with an abelian group. Recently, several characterizations of a parallelogram space were given in [1],[5],[4] and the paper presented here adds a new one. Namely, for a parallelogram space we give a "geometrically" obvious definition of a symmetry and prove that the symmetries are involutory mappings (including the identity or not) transitive on "the points" which satisfy the three symmetries theorem (TST). Hence we have a motivation to introduce the notion of a TST-space which is defined as an ordered pair of a nonvoid set and its mappings (symmetries) satisfying the above mentioned properties of symmetries in a parallelogram space. It will be proved that for every TST-space there is a parallelogram space in which the symmetries coincide with the symmetries of the given TST-space. Moreover, every TST-space is a TST-space induced by an abelian group.

Definition 1. ([2]) A parallelogram space (Q,P) is a nonvoid set Q with a quaternary relation $P \subseteq Q^4$ such that the following conditions are satisfied:

(P1) $(a,b,c,d) \in P$ implies $(a,c,b,d) \in P$, for all $a,b,c,d \in Q$

(P2) $(a,b,c,d) \in P$ implies $(c,d,a,b) \in P$, for all $a,b,c,d \in Q$

(P3) $(a,b,x,y),(x,y,c,d) \in P$ implies $(a,b,c,d) \in P$, for all $a,b,c,d,x,y \in P$;

(P4) For any three $a,b,c \in Q$ there is exactly one element $d \in Q$ such that $(a,b,c,d) \in P$.

In the following lemma we prove some elementary properties of a parallelogram space which will be used afterwards.

Lemma 1. If (Q,P) is a parallelogram space, then

1° $(a,b,c,d) \in P$ implies $(b,a,d,c),(d,b,c,a) \in P$, for all $a,b,c,d \in Q$;

2° $(x,a,b,y),(x,c,d,y) \in P$ implies $(a,c,d,b) \in P$, for all $a,b,c,d,x,y \in Q$;

3° (a,x,y,b), $(c,x,y,d) \in P$ implies $(a,c,d,b) \in P$, for all $a,b,c,d,x,y \in Q$;

Proof. 1° From $(a,b,c,d) \in P$, it follows $(a,c,b,d) \in P$ by (P1), and therefore $(b,d,a,c) \in P$ by (P2) which gives $(b,a,d,c) \in P$ by (P1). Since $(a,b,c,d) \in P$ implies $(b,d,a,c) \in P$, it follows $(d,b,c,a) \in P$ by the just proved implication.

2° Let be $(x,a,b,y),(x,c,d,y) \in P$ and define $w \in Q$ by $(a,y,c,w) \in P$. Then (x,b,a,y), $(a,y,c,w) \in P$ implies

$(x,b,c,w) \in P$ by (P3) and we have $(x,c,b,w) \in P$. Now, from $(b,w,x,c),(x,c,d,y) \in P$ it follows $(b,w,d,y) \in P$. i.e. $(b,d,w,y) \in P$. Since $(a,c,y,w),(y,w,d,b) \in P$ we obtain $(a,c,d,b) \in P$.

3° Let be $(a,x,y,b),(c,x,y,d) \in P$ and hence (y,b,a,x), $(y,d,c,x) \in P$. It follows $(b,d,c,a) \in P$ by 2° and therefore $(a,c,d,b) \in P$ by 1°

Let (Q,P) be a parallelogram space. For any $(a,b) \in Q^2$ we define a <u>symmetry</u> $f_{ab}: Q \to Q$ by the following equivalence

$$(*) \qquad (\forall\, a,b,x,y \in Q) \quad f_{ab}(x) = y \Leftrightarrow (x,a,b,y) \in P.$$

The mapping f_{ab} is well defined for all $a,b \in Q$ because of (P4) and the set of all symmetries of a parallelogram space (Q,P) we denote by $S(Q,P)$, briefly S.

Since $(a,a,b,b) \in P$ holds for all $a,b \in Q$, because of (P2), (P3), it follows $f_{ab}(a) = b$.

<u>Proposition 1.</u> If (Q,P) is a parallelogram space and S the corresponding set of symmetries, then the following is valid:

1° $f^2 = 1$ for all $f \in S$, where 1 is the identity on Q;

2° For any two $x,y \in Q$, there is a symmetry $f \in S$ such that $f(x) = y$.

Proof. Since $(x,a,b,y) \in P$ iff $(y,a,b,x) \in P$ by Lemma 1, it follows 1° Further, since $(x,a,b,y) \in P$ iff $(b,y,x,a) \in P$ by (P2), it follows 2°

<u>Proposition 2.</u> (The three symmetries theorem). Let (Q,P) be a parallelogram space and S the set of its symmetries. Then for any three symmetries $f_1,f_2,f_3 \in S$, their composition $f_1f_2f_3$ is a symmetry, i.e. $f_1f_2f_3 \in S$.

Proof. Let be $f_1 = f_{a_1a_2}, f_2=f_{b_1b_2}, f_3=f_{c_1c_2}$ for some $a_1,a_2,b_1,b_2,c_1c_2 \in Q$ and for a fixed element $o \in Q$ we define $a,b,c,d \in Q$ by $f_1(o) = a$, $f_2(o) = b$, $f_3(o) = c$, $(b,a,c,d) \in P$ i.e. $(a,b,d,c) \in P$. Now, for any $x \in Q$, let be $f_1(x) = y$, $f_2(y) = z$, $f_3(z) = w$. Since $(x,a_1,a_2,y),(o,a_1,a_2,a) \in P$ it follows $(x,o,a,y) \in P$ by Lemma 1. Similarly, we have (y,o,b,z) $(z,o,c,w) \in P$. Then $(y,o,a,x),(y,o,b,z) \in P$ implies $(a,x,b,z) \in P$ i.e. $(a,b,x,z) \in P$. Since $(a,b,d,c) \in P$, it follows $(x,z,d,c) \in P$. Further, $(x,d,z,c),(z,c,o,w) \in P$ implies $(x,d,o,w) \in P$. Hence $f_{do}(x) = w = f_1f_2f_3(x)$ for all $x \in Q$ and therefore $f_{do}=f_1f_2f_3$ which proves the proposition.

Remark 1. For a fixed element $o \in Q$ and any $a \in Q$, let the symmetry f_{ao} be denoted by f_a. Then the following equivalence is valid:

$$(**) \qquad (\forall\, a,b,c,d \in Q)\ f_a(b) = f_c(d) \Leftrightarrow (a,b,c,d) \in P.$$

Indeed, let be $f_a(b) = u$ which is equivalent to $(b,a,o,u) \in P$, i.e. $(a,b,u,o) \in P$. Then $(a,b,c,d) \in P$ iff $(c,d,u,o) \in P$ iff $(d,c,o,u) \in P$ iff $f_c(d) = u$ and $(**)$ is proved. Further, $f_a(0)=a$ for all $a \in Q$ and for any symmetry $f \in S$ there is exactly one element $a \in Q$ such that $f = f_a$. Namely, let be $f = f_{a_1a_2}$ and define $a \in Q$ by $f(o) = a$, i.e. $(o,a_1,a_2,a) \in P$. Then, taking into account Lemma 1, $(x,a_1,a_2,y) \in P$ iff $(x,o,a,y) \in P$, i.e. $f(x) = y$ iff $f_a(x) = y$ and hence $f = f_a$.

Propositions 1 and 2 give us a motivation to introduce a notion of a TST-space ("The three symmetries theorem" - space) for which we shall show that it is related to a parallelogram space.

Definition 2. A TST-space (Q,S) is a nonvoid set Q with a set S of its mappings which are called symmetries such that the following conditions are satisfied:

(S1) $f^2 = 1$ for all $f \in S$, where 1 is the identity on Q;

(S2) for any two $x,y \in Q$, there is a symmetry $f \in S$ such that $f(x) = y$;

(S3) $f_1f_2f_3 \in S$, for all $f_1,f_2,f_3 \in S$.

Hence, a TST-space is an ordered pair (Q,S) where Q is a nonvoid set and S a set of its involutory mappings (including the identity or not) which is transitive on Q and for which the three symmetries theorem holds.

As a direct consequence of the propositions 1. and 2. we obtain the following proposition.

Proposition 3. Let (Q,P) be a parallelogram space and S=S(Q,P) the set of its symmetries, i.e. the set of all mappings f_{ab} which are defined by (*). Then (Q,S) is a TST-space.

Our aim is to show that for every TST-space (Q,S) there is a parallelogram space (Q,P) such that S=S(Q,P) is valid.

Firstly, we shall prove the following two lemmas (cf. [3]).

Lemma 2. Let (Q,S) be a TST-space and $f_1,f_2,f_3,f_4 \in S$ any four symmetries.

1° If $f_1(x) = f_2(x)$ for some $x \in Q$, then it follows $f_1=f_2$;

2° If $f_1f_2(x) = f_3f_4(x)$ for some $x \in Q$, then it follows $f_1f_2=f_3f_4$.

Proof. 1° Let be $f_1(x) = f_2(x)$ for some $x \in Q$ and $f_1,f_2 \in S$. For an arbitrary element $y \in Q$, let f be the symmetry for which $f(x) = y$. Applying the equality $g_1g_2g_3=g_3g_2g_1$, which holds for all $g_1,g_2,g_3 \in S$ because of (S3), (S1), it follows

$$f_1(y) = f_1f(x) = f_1ff_2f_1(x) = f_1f_1f_2f(x) = f_1f(x) = f_2(y)$$

and therefore $f_1 = f_2$.

2° Let be $f_1f_2(x) = f_3f_4(x)$ for some $x \in Q$ and $f_1, f_2, f_3, f_4 \in S$. Then $ff_1f_2(x) = ff_3f_4(x)$ holds for any $f \in S$. Since $ff_1f_2, ff_3f_4 \in S$ it follows $ff_1f_2 = ff_3f_4$ by 1° Hence $f_1f_2 = f_3f_4$.

Remark 2. Because of the previous lemma, for any two $x,y \in Q$ there is a unique symmetry $f \in S$ such that $f(x) = y$.

Lemma 3. Let (Q,S) be a TST-space and for any two $a,b \in Q$ let f_{ab} denote a uniquely determined symmetry in S which maps a to b. Then

1° $f_{ab} = f_{ba}$, for all $a,b \in Q$;

2° $f_{ax} = f_{cy}$ and $f_{by} = f_{dx}$ implies $f_{ab} = f_{cd}$ for all $a,b,c,d,x,y \in Q$.

Proof. Since $f(a) = b$ implies $f(b) = a$, the statement 1° is valid. Now, if $f_{ax} = f_{cy}$ and $f_{by} = f_{dx}$, then

$$f_{cd}(a) = f_{cd}f_{ax}(x) = f_{cd}f_{ax}f_{dx}(d) = f_{dx}f_{ax}f_{cd}(d) = f_{dx}f_{ax}(c) =$$
$$= f_{by}f_{cy}(c) = b$$

and therefore $f_{cd} = f_{ab}$.

If we define f_{ab} as it was done in the previous lemma, then we can construct a parallelogram space starting from a TST-space. So we obtain an additional characterization of a parallelogram space (cf. [2],[1],[5],[4]).

Proposition 4. Let (Q,S) be a TST-space and $P \subseteq Q^4$ a quaternary relation defined by (*) where f_{ab} denotes the symmetry which maps a to b. Then (Q,P) is a parallelogram space.

Proof. Let be $(a,b,c,d) \in P$ i.e. $f_{bc}(a) = d$. Then it follows $f_{cb}(a) = d$ by Lemma 3, i.e. (P1) holds. Moreover, $(a,b,c,d) \in P$

implies $f_{bc} = f_{ad}$ and therefore $f_{da}(c)=f_{bc}(c)=b$, i.e. $(c.d,a,b) \varepsilon P$. Now, let be $(a,b,x,y),(x,y,c,d) \varepsilon P$ i.e. $f_{bx}(a)=y$, $f_{yc}(x) = d$ and therefore $f_{ay}=f_{bx}, f_{dx} = f_{cy}$ which implies $f_{ad} = f_{bc}$. Hence $f_{bc}(a) = d$, i.e. $(a,b,c,d) \varepsilon P$.

The condition (P4) is evidently satisfied and the proposition is proved.

Remark 3. If $o \varepsilon Q$ is a fixed element in a TST-space (Q,S) and f_a denotes the symmetry f_{ao}, for all $a \varepsilon Q$, then the equivalence (**) is valid. Indeed, if $f_a(b) = f_c(d)$ i.e. $f_{ao}(b) = f_{co}(d)$ then

$$f_{bc}(a) = f_{bc}f_{ao}f_{co}(c) = f_{co}f_{ao}f_{bc}(c) = f_{co}f_{ao}(b) = d$$

i.e. $(a,b,c,d) \varepsilon P$. Conversely, if $(a,b,c,d) \varepsilon P$, i.e. $f_{bc}(a)=d$ then

$$f_{ao}(b) = f_{ao}f_{bc}f_{co}(o) = f_{co}f_{bc}f_{ao}(o)=f_{co}(d)$$

i.e. $f_a(b) = f_c(d)$.

Now, let $(Q,+)$ be an abelian group. For every $a \varepsilon Q$, define a mapping $f_a: Q \to Q$ by $f_a(x) = a-x$, for all $x \varepsilon Q$. If S_+ denotes the set of all so define f_a then it is easy to verify that (Q,S_+) is a TST-space. In the following proposition we shall show that any TST-space is of that form, i.e. for every TST-space (Q,S) there is an abelian group $(Q,+)$ such that $S = S_+$.

Proposition 5. Let (Q,S) be a TST-space, $o \varepsilon Q$ a fixed element and f_a denotes uniquely determined symmetry which maps o to a, $a \varepsilon Q$. If $+$ is a binary operation on Q defined by the equality

$$f_{a+b} = f_a f_o f_b$$

for all $a,b \varepsilon Q$, then $(Q,+)$ is an abelian group, o its unity and

$$f_a(x) = a-x$$

for all $a,x \in Q$.

Proof. Let be underlined that for all $a \in Q$, a symmetry f_a is uniquely determined by (S2) and that for all $f \in S$ there is a unique $a \in Q$ such that $f = f_a$. Since $f_a f_o f_b$ is a symmetry, for all $a,b \in Q$, it follows that $(Q,+)$ is a groupoid and it is commutative because of $f_a f_o f_b = f_b f_o f_a$, which holds by (S1). Obviously, $(Q,+)$ is an associative quasigroup, hence, an abelian group. The element o is the unity in $(Q,+)$ and $-a$ the inverse of a, $a \in Q$, is given by $f_{-a}=f_o f_a f_o$. Hence, for all $a,x \in Q$

$$f_a(x) = f_a f_x(o) = f_a f_o f_o f_x f_o(o) = f_{a-x}(o) = a-x.$$

Corollary 2 . If (Q,P) is a parallelogram space then there is an abelian group $(Q,+)$ such that

$$(***) \qquad (\forall\, a,b,c,d \in Q)\ (a,b,c,d) \in P \Leftrightarrow a-b = c-d.$$

Proof. The statement follows immediately from Proposition 3 and Remark 1.

Remark 4. Of course, if $(Q,+)$ is an abelian group and the quaternary relation P on Q is defined by (***), then (Q,P) is a parallelogram space.

Remark 5. If (Q,S) is a TST-space, then it is easy to prove that the group generated by the compositions of even number of symmetries is abelian and isomorphic to an abelian group $(Q,+)$ defined in the Proposition 5 (every composition of an even number of symmetries can be written as $f_a f_b$, for some $a,b \in Q$, and $f_a f_b \to a-b$ determinates an isomorphism).

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