STOCHASTICA, Vol. VIII, Nº 3, 1984

A SHORT NOTE ON REPRESENTATION OF L-FUZZY SETS BY MOORE'S FAMILIES

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Let E be a nonvoid set and $(L,\vee,\wedge)$ a lattice. An L-FUZZY SET A is any map $\mu_A:E \to L$ (Goguen [1]). The case $L=[0,1]$ has the usual definition of a fuzzy set (Zadeh [2]). Let $\mathcal{L}(E,L)$ denote the class of all L-fuzzy sets over E. Being L a bounded lattice (with zero 0 and unit 1) a classical set can be interpreted as your characteristic function and can be considered $\mathcal{P}(E)\subset \mathcal{L}(E,L)$.

The operations $\vee$, $\wedge$ induce operations $\cup$ and $\cap$ in $\mathcal{L}(E,L)$. $(\mathcal{L}(E,L),\cup,\cap)$ has the same structure as $(L,\vee,\wedge)$.

For every L-fuzzy set $A\in \mathcal{L}(E,L)$ we shall consider the ordinary subsets of E, so called α-cuts of A,

$$A_\alpha=\{x\,|\,x\in E,\ \mu_A(x)\geqslant \alpha\}\in \mathcal{P}(E)$$

$$A^\beta=\{x\,|\,x\in E,\ \mu_A(x)\leqslant \beta\}\in \mathcal{P}(E)$$

and the families $\mathcal{F}_A=\{A_\alpha\}_{\alpha\in L}$ $\mathcal{G}_A=\{A^\beta\}_{\beta\in L}$, with well known properties. Among them we shall emphasize that

a) $\mathcal{F}_A$ and $\mathcal{G}_A$ characterize the L-fuzzy set A, that is to say

$$A_\alpha= B_\alpha\ \forall\alpha\in L \Rightarrow A=B \quad\text{and}\quad A^\beta=B^\beta\ \forall\beta\in L \Rightarrow A=B.$$

b) If L is a bounded lattice then $A_0 = A^1 = E \ \forall A \in \mathcal{L}(E,L)$ and if L is a complete lattice and H is a nonvoid subset of L then

$$\bigcap_{\alpha \in H} A_\alpha = A_{\sup H},\ \bigcap_{\beta \in H} A^\beta = A^{\inf H}.$$

Generaly a family $\mathcal{F} \subseteq \mathcal{P}(E)$ is a Moore's Family (Dubreil, Dubreil-Jacotin [3]) if $E \in \mathcal{F}$ and $\bigcap_{M \in \mathcal{F}'} M \in \mathcal{F} \ \ \forall \mathcal{F}' \subseteq \mathcal{F}$. According to b), if L is a complete lattice (therefore a bounded lattice) then $\mathcal{F}_A$ and $\mathcal{G}_A$ are Moore's Families.

Let $A \in \mathcal{L}(E,L)$ and $x \in E$. The ordinary subset $T_x = \{\alpha \in L \mid x \in A_\alpha\} = \{\alpha \in L \mid \mu_A(x) \geqslant \alpha\} = (\mu_A(x)] \subseteq L$ is the principal ideal of L generated by $\mu_A(x)$. It may then be

$$\mu_A(x) = \sup T_x \in T_x \qquad (1)$$

Similarly $T^x = \{\beta \in L \mid x \in A^\beta\}$ is the dual ideal (filter) generated by $\mu_A(x)$ and

$$\mu_A(x) = \inf T^x \in T^x \qquad (1')$$

(1) and (1') are representations of the L-fuzzy set A. Being L a bounded lattice, the next lemma shows that (1) and (1') can be stated by means of the families $\{A_\alpha\}_{\alpha \in L}$ and $\{A^\beta\}_{\beta \in L}$.

<u>Lemma 1.</u> If L is a bounded lattice and $A \in \mathcal{L}(E,L)$ then

1) $T_x = \{\inf\{\alpha, f_{A_\alpha}(x)\} \mid \alpha \in L\} \quad \forall x \in E$, where f_{A_α} is the characteristic function of $A_\alpha \subseteq E$.

2) $T^x = \{\sup\{\beta, f_{(A^\beta)^c}(x)\} \mid \beta \in L\} \ \forall x \in E$, where $f_{(A^\beta)^c}$ is the characteristic function of $(A^\beta)^c$ (complement of A^β).

Proof 1) If $\alpha \in T_x$ then $x \in A_\alpha$ and $f_{A_\alpha}(x) = 1$ and $\inf\{\alpha, f_{A_\alpha}(x)\} = \alpha$ showing that $T_x \subseteq \{\inf\{\alpha, f_{A_\alpha}(x)\} \mid \alpha \in L\}$. If $\gamma \in L$

then $\inf\{\gamma, f_{A_\gamma}(x)\} \in \{0,\gamma\}$. If it is 0 then is belongs to the ideal T_x, if it is γ then $x \in A_\gamma$ showing that $\{\inf\{\alpha, f_{A_\alpha}(x)\} | \alpha \in L\} \subseteq T_x$.

2) Using the duality principle we infer 2).

Now let $\underset{\sim}{\alpha}$ be the L-fuzzy set $\mu_{\underset{\sim}{\alpha}}(x)=\alpha \ \forall \ x \in E$, we have the

Corollary 1. If L is a bounded lattice and $A \in \mathcal{L}(E,L)$ then

$$\text{a)} \quad A = \cup(\underset{\sim}{\alpha} \cap A_\alpha)$$

$$\text{b)} \quad A = \cap(\underset{\sim}{\beta} \cup (A^\beta)^c)$$

where A_α and $(A^\beta)^c$, subsets of E, are viewed as L-fuzzy sets and $\cup$, $\cap$ are operations in $\mathcal{L}(E,L)$.

Proof. It is trivial from lemma 1.

Note that the known formula of the membership function of a fuzzy set A (where L=[0,1]) is a particular case of the corollary 1 a). Observe however that in corollary 1, L need not be a complete lattice. The formulas (1) and (1') are representations of an L-fuzzy set A using the families $\mathcal{F}_A$ and $\mathcal{G}_A$. Now we shall analyse the next question, to establish L-fuzzy sets using certain families of subsets.

Theorem 1. Let θ be verifying

1) $\theta: L \to \mathcal{P}(E)$
2) $\theta(\alpha) \cap \theta(\beta)=\theta(\alpha \vee \beta) \quad \forall(\alpha,\beta) \in L \times L$
3) For all $x \in E$ the subset $R_x=\{\alpha | \alpha \in L,\ x \in \theta(\alpha)\}$ has maximum.

Then R_x is an ideal and $\mu_A(x)=\max R_x \quad \forall x \in E$ is an L-fuzzy set A with $A_\alpha=\theta(\alpha) \quad \forall \alpha \in L$.

Proof. $R_x \neq \emptyset$ by 2). If $\alpha,\beta \in R_x$ and $\gamma \in L$ then $x \in \theta(\alpha)$ and $x \in \theta(\beta)$ and so $x \in \theta(\alpha) \cap \theta(\beta)=\theta(\alpha \vee \beta)$ that is $\alpha \vee \beta \in R_x$. Since

$\alpha=\alpha\vee(\alpha\wedge\gamma)$ we conclude that $\theta(\alpha)=\theta(\alpha)\cap\theta(\alpha\wedge\gamma)$, that is $\theta(\alpha)\subseteq\theta(\alpha\wedge\gamma)$ proving that $\alpha\wedge\gamma\in R_x$. We conclude that R_x is an ideal.

Let A be the L-fuzzy set $\mu_A(x)=\max R_x$ $\forall x\in E$ and let A_α be the subset $A_\alpha=\{x|\mu_A(x)\geqslant\alpha\}$. If $x\in A_\alpha$ then $\alpha\in R_x$, because R_x is an ideal and $\mu_A(x)\in R_x$, and we conclude that $x\in\theta(\alpha)$. Conversely if $x\in\theta(\alpha)$ then $\alpha\in R_x$, that is $\mu_A(x)\geqslant\alpha$ proving that $x\in A_\alpha$.

The theorem 1 has the following dual statement

Theorem 1'. Let θ verifying

1) $\theta: L\to P(E)$
2) $\theta(\alpha)\cap\theta(\beta)=\theta(\alpha\wedge\beta)$ $\forall(\alpha,\beta)\in L\times L$.
3) For all $x\in E$ the subset $R^x=\{\beta|\beta\in L,\ x\in\theta(\beta)\}$ has minimum.

The R^x is a dual ideal (filter) and $\mu_A(x)=\min R^x$ $\forall x\in E$ is an L-fuzzy set A with $A^\beta=\theta(\beta)$ $\forall\beta\in L$.

The next example shows that it can not be generally inferred from theorem 1 that $\bigcap_{\alpha\in H}\theta(\alpha)=\theta(\sup H)$ $\forall H\subseteq L$, $H\neq\emptyset$.

Example 1. Let E be the interval $[3,\pi)\subset R$ and L the sequence of rational numbers $(a_n)=(3,3.1,3.14,3.141,3.1415,\ldots)$ i.e. $a_n=\frac{[10^n\pi]}{10^n}$ where $n=0,1,2,\ldots$ and $[x]$ is the entire part of x. L is not a complete lattice ($\nexists$ sup L). We shall consider $\theta(\alpha)=[\alpha,\pi)$ $\forall\alpha\in L$. θ satisfies 1) 2) and 3) of the theorem 1. Observe that $\theta(\sup L)$ lacks any meaning and $\bigcap_{\alpha\in L}\theta(\alpha)=\emptyset$.

In this way, if L is a complete lattice, theorem 1 can be stated as follows.

Theorem 2. Let L be a complete lattice and let θ be a Moore's Family in E, $\theta:L\to P(E)$ verifying

$$\bigcap_{\alpha \in H} \theta(\alpha)=\theta(\sup H) \quad \forall H \subseteq L, \quad H \neq \emptyset$$

then $\mu_A(x)=\sup\{\alpha | \alpha \in L, \ x \in \theta(\alpha)\}$ is an L-fuzzy set $A \in \mathcal{L}(E,L)$ whit $A_\alpha=\theta(\alpha) \quad \forall \alpha \in L$.

Proof. It suffices to show that $\mu_A(x) \in R_x$. From $\bigcap_{\alpha \in R_x} \theta(\alpha)=\theta(\sup R_x)=\theta(\mu_A(x))$ it is obvious that $\mu_A(x) \in R_x$.

Similarly, the theorem 2 has a dual statement.

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