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CLASSIFICATION OF THE REGULAR DE MORGAN ALGEBRAS OF FUZZY SETS

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ABSTRACT

A characterization of regular lattices of Fuzzy Sets and their isomorphisms are given in part I. A characterization of involutions on regular lattices of Fuzzy Sets and the isomorphisms of De Morgan Algebras of Fuzzy Sets are given in Part II. Finally all classes of De Morgan Algebras of Fuzzy Sets respect to isomorphisms are completely described.

Introduction.

The aim of this work is to characterize the classes of regular De Morgan algebras of fuzzy sets with respect to isomorphisms, which is useful both from the point of view of the structural analysis of these algebras and in order to continue the work about the representation of De Morgan algebras by fuzzy sets ([3], [4], [5] and [6]).

For these reasons we study in the first part regular lattices of fuzzy sets and theirs isomorphisms from the results obtained in [2] and [3]. On the other hand, the study of involutions that may be defined in these lattices made in [9], allow to characterize the isomophisms between De Morgan algebras of fuzzy sets. The work conclude with a classification of these algebras with respect to isomorphisms, by continuing the study begin in [2] and [3].

We will denote by L(X) the lattice $(\underset{\sim}{P}(X),\cap,\cup)$, by C the classical complementation that endows P(X) of Boole algebra structure and by $\bar{A}$ the complement by C of all A of P(X). We will call <u>singleton</u> and we will denote by δ_x^α, the fuzzy set definite by $\delta_x^\alpha(a)=0$ if $a\neq x$ and $\delta_x^\alpha(x)=\alpha$ (if $\alpha=1$ we obtain the singletons of P(X)). We will denote by $\bar{\delta}_x^\alpha$ the fuzzy set definite by $\bar{\delta}_x^\alpha(a)=1$ if $a\neq x$ and $\bar{\delta}_x^\alpha(x)=\alpha$. Finally, we will denote by K the set of all subsets of the unit interval that contain {0,1}, that is $K=\{J;\ \{0,1\}\subset J\subset[0,1]\}$.

I. Isomorphisms of Regular lattices of Fuzzy Sets.

I-1. <u>Regular Lattices of Fuzzy Sets.</u>

<u>Definition 1.</u> A sublattice S of L(X) is <u>regular</u> if it contains P(X).

Given a family $R=\{J_x\in K;\ x\in X\}$ if we denote by:

$$P_R(X) = \{A\in \underset{\sim}{P}(X);\ A(x)\in J_x \text{ for all } x\in X\}$$

and

$$P'_R(X) = \{A\in P_R(X);\ A(x)\notin\{0,1\} \text{ is finite} ,$$

then $P_R(X)$ and $P'_R(X)$ are both regular sublattices of L(X).

Regular sublattices of L(X) were characterize in [3] by the following theorem.

<u>Theorem 1.</u> A sublattice S of L(X) is regular if, and only if, there exists a family $R=\{J_x\in K,;\ x\in X\}$ such that $P'_R(X)\subset S\subset P_R(X)$.

So, it is clear that $P_R(X)$ is the greatest sublattice with values in R, we will call it <u>sublattice R-maximal.</u> On the other hand it is easy to prove that the singletons of a regular sublattice of L(X) are the same that the singletons of the R-maximal lattice that contains it.

I-2. <u>Characteritation of Isomorphisms.</u>

<u>Proposition 1.</u> Let H be an isomorphism between regular sublattices of L(X), $H|_{P(X)}$ is an automorphism of P(X)■

So, if H is an isomorphism between regular sublattices of L(X), $H|_{P(X)}$ will be of the form $(H|_{P(X)})(A)=A\circ\sigma$, where σ is a permutation of X.

<u>Proposition 2.</u> Any isomorphism H between S and S' regular sublattices of L(X) is univocally determined by the images of the singletons of S.

Proof. It is obvious as any isomorphism is morphism respect to the infinite unions and intersections and for any A of S, $A=\bigcup_{x\in X}\delta_x^{A(x)}$■

As singletons of a regular sublattice are the same that the singletons of R-maximal lattice that contain it, all isomorphism between regular sublattices will be the restriction of an isomorphisms between R-maximal lattices. It is for this reason that we only study isomorphisms between R-maximals lattices.

<u>Theorem 2.</u> Let $R=\{J_x;\ x\in X\}$ and $R'=\{J'_x;\ x\in X\}$ be two families of elements of K, then H is an isomophism between $(P_R(X),\cap,\cup)$ and $(P_{R'}(X),\cap,\cup)$ if, and only if, there exists a permutation σ of X and a family $\{f_x:J_x\to J'_{\sigma(x)};\ x\in X\}$ of increasing bijections such that for any A of $P_R(X)$, $H(A)=\bigcup_{x\in X}\delta_{\sigma(x)}^{f_x(A(x))}$.

Proof. If H is an isomorphism between $P_R(X)$ and $P_{R'}(X)$ we know that $H|_{P(X)}$ is an automorphism of $P(X)$, then there exists σ, permutation of X, such that for any A of $P(X)$, $H(A)=A\circ\sigma$; in particular $H(\delta_x) = \delta_{\sigma(x)}$. If we define

$[\emptyset, \delta_x]_{J_x} = \{A \in P_R(X); \emptyset \subset A \subset \delta_x\}$ and

$[\emptyset, \delta_{\sigma(x)}]_{J'_{\sigma(x)}} = \{A \in P_{R'}(X); \emptyset \subset A \subset \delta_{\sigma(x)}\}$ it is clear

that:

$$H([\emptyset, \delta_x]_{J_x}) = [\emptyset, \delta_{\sigma(x)}]_{J'_{\sigma(x)}} \qquad (1)$$

For any $x \in X$ we define $f_x : J_x \to J'_{\sigma(x)}$ by

$$f_x(\alpha) = \beta \quad \text{if} \quad H(\delta_x^{\alpha}) = \delta_{\sigma(x)}^{\beta} .$$

All f_x are increasing bijections as H is an isomorphism and (1) holds.

On the other hand, for any $A \in P_R(X)$, $H(A) = \bigcup_{x \in X} H(\delta_x^{A(x)})$ $= \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}$.

Reciprocally, if we define $H(A) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}$, for any

$A \in P_R(X)$, where σ is a permutation of X and $\{f_x : J_x \to J'_{\sigma(x)}; x \in X\}$ is a family of increasing bijections, it is immediate that H is one to one increasing maping, and a morphism with respect to the union. Let us see that H is morphism with respect to the intersection.

$$H(A) \cap H(B) = \Big(\bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}\Big) \cap \Big(\bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(B(x))}\Big) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x((A \cap B)(x))} = H(A \cap B) \blacksquare$$

In the particular case that any $J_x \in R$ is equal to $J \in K$, we will denote by $P_J(X)$ the R-maximal sublattice of $\underset{\sim}{P}(X)$.

Definition 2. For any permutation $\sigma \in X$ the automorphism generated by σ, is the automorphism of $P_J(X)$ defined by $H_\sigma(A) = A \circ \sigma$, for any A of $P_J(X)$. We denote it by H_σ.

Definition 3. We say that an isomorphism F from $P_J(X)$ to $P_{J'}(X)$ is pontwise functionally expreseble (p.f.e.) if there exists a family $\{f_x : J \to J';\ x \in X\}$ of increasing bijections such that for any A of $P_J(X)$: $(F(A))(x) = f_x(A(x))$.

Using the previous two definition we obtain the following:

Corollary 1. Any isomorphism between $P_J(X)$ and $P_{J'}(X)$ is obtained by compositing an automorphism generated by a permutation with an isomorphism p.f.e.

Proof. If H is an isomorphism between $P_J(X)$ and $P_{J'}(X)$, because of the previous theorem we know that

$$H(A) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}$$, being σ a permutation of X, and

$\{f_x : J \to J';\ x \in X\}$ a family of increasing bijections. Let H_σ be the automorphism of $P_J(X)$ generated by σ and F the isomorphism from $P_J(X)$ to $P_{J'}(X)$ p.f.e. by the family $\{f_x : J \to J';\ x \in X\}$, that is, $(F(A))(x) = f_x(A(x))$, for any A of $P_J(X)$.

$H_\sigma \circ F$ is an isomorphism from $P_J(X)$ to $P_{J'}(X)$ as it is a composition of isomorphisms.

Let us see that $H = H_\sigma \circ F$. We only need to prove that H and $H_\sigma \circ F$ coincide over the singletons of $P_J(X)$, and this is easy because for any $x \in X$ and $\alpha \in J$ we have:

$$(H_\sigma \circ F)(\delta_x^\alpha) = H_\sigma(\delta_x^{f_x(\alpha)}) = \delta_{\sigma(x)}^{f_x(\alpha)} = H(\delta_x^\alpha) \blacksquare$$

In the particular case $J = J' = [0,1]$ we have again the characterization of isomorphisms of L(X) given in [2].

II. Isomorphisms of Regular De Morgan Algebras of Fuzzy Sets.

II.1. Negations in Regular Lattices of Fuzzy Sets.

In this part, whenever we talk of sublattices of L(X), we assume that they are regular sublattices of L(X).

Definition 4. Let S be a ordered set with element maximum and minimum. A mapping n from S to S is a strong negation or involution if it is a decreasing bijection, and $n^2=j$, where j means the identity in S.

In the case that S ∈ K we call it strong negation function.

We know that a De Morgan algebra of fuzzy sets is a lattice of fuzzy sets with a strong negation. In [13], they were studied and characteraized all strong negations defined in lattices of fuzzy sets. The following results were obtained.

Proposition 3. If n is a strong negation of S, sublattice of L(X), n is extention of a strong negation of P(X).

It is known that all strong negations of P(X) are of the type $C \circ H_\sigma$, where H_σ is the automorphism generated by σ, permutation of X such that $\sigma^2=j$.

Let us define two kinds of strong negations:

Definition 5. Let S be a sublattice of L(X) and n a strong negation of S, we say that:

n fulfil The Extension Principle (E.P.) if $n|_{P(X)} = C$

n fulfil The Generalized Extension Principle (G.E.P.) if $n|_{P(X)} = C \circ H_\sigma$ where $\sigma \neq j$.

Proposition 4. Any strong negation defined in a regular lattice is restriction of a negation of the R-maximal lattice that contains it.

Proof. Any strong negation defined in S, sublattice of L(X), is univocally determined by the images of singletons of S, as all strong negations fulfil the De Morgan laws.

The proposition is evident since the singletons of S and singletons of the R-maximallattice that contain S, are the same. ■

For these reason we only study negations in R-maximals lattices.

Definition 6. We say that a strong negation n of the R-maximal lattice is pointwise functionally expressible (p.f.e.) if there exist a family $\{n_x : J_x \to J_x;\ x \in X \text{ and } J_x \in R\}$ of strong negations functions, such that for all $x \in X$ and for all A of $P_R(X)$, $(nA)(x) = n_x(A(x))$.

Theorem 3. A strong negation $n \in P_R(X)$ fulfils the E.P. if, and only if, n is p.f.e.

Definition 7. Let J_1, J_2, be two elements of K, we call strong Galois correspondance of J_1, J_2, to any pair of mappings (n_{12}, n_{21}) $n_{12}: J_1 \to J_2$ and $n_{21}: J_2 \to J_1$, satisfying:

i) n_{12}, n_{21} decreasing and one-one mappings.

ii) $n_{21} \circ n_{12} = j|_{J_1}$, and $n_{12} \circ n_{21} = j|_{J_2}$.

Definition 8. A strong negation n of $P_R(X)$ is generated by a family of strong Galois correspondances if there exist an involutive permutation of X, denoted by S and a family of strong Galois correspondances $\{(n_{xS(x)}, n_{S(x)x});\ n_{xS(x)}: J_x \to J_{S(x)}$, and $n_{S(x)x}: J_{S(x)} \to J_x$, for every $x \in X\}$, such that

$$[nA](x) = n_{S(x)x}(A(S(x))), \text{ for every } A \text{ of } P_R(X).$$

Theorem 4. A strong negation $n \in P_R(X)$ satisfies the G.E.P. if, and only if, n is generated by a family of strong Galois correspondances.

In the particular case that $J_x = J_{S(x)}$, $n_{xS(x)} = n_{S(x)x}$; then what we have is a strong negation of J_x.

Corollary 2. A strong negation n of $P_J(X)$ satisfies the G.E.P. if, and only if, $n = n' \circ H_S$, where H_S is the automorphism generated by S, (an involutive permutation of X), and n' is a strong negation of $P_{J(X)}$ generated by the family $\{n_x : J \to J,\ x \in X\}$ of strong negation functions with the condition that $n_x = n_{S(x)}$.

As a consequence of this corollary we obtain the characterization of the strong negations of L(X) that fulfil the G.E.P. given in [1].

Before facing the problem of isomorphisms between De Morgan algebras of fuzzy sets, we need some results concerning to strong negation functions and strong Galois correspondances.

Remember ([14]) that if J,T are two elements of K and f an increasing one-one mapping from J to T, then $n = f^{-1} o (1-j)$ of is a strong negation function of J, and f is called an <u>additive generator</u> of n.

Proposition 5. A mapping n from J to J is a strong negation function if, and only if, n has an additive generator.

The other hand, in [7] next results are given:

Definition 9. Let n and n' be two strong negation functions of J and J' respectively, n <u>is equivalent</u> to n' if there exist, an increasing one-one mapping f from J to J' such that fon = n'of.

Proposition 6. Two strong negation functions are equivalents if, and only if, there exist an increasing one-one mapping from J to J' and n, n' have the same number of fixed points (one or none).

With respect to strong Galois correspondances, remember that given J_1, J_2 and T, elements of K, f an increasing one-one mapping from J_1 to T and g an increasing one-one mapping from J_2 to 1-T, and if we define $n_{12}=g^{-1}o(1-j)$ of and $n_{21}=f^{-1}o(1-j)og$, then (n_{12},n_{21}) is a strong Galois correspondance between J_1 and J_2. Then (f,g) is called the generator pair of the strong Galois correspondance (n_{12},n_{21}). In [12] the following characterisation of Galois correspondance is given.

Proposition 7. A pair (n_{12},n_{21}) of mappings from J_1 to J_2 and from J_2 to J_1 respectivelly, is a strong Galois correspondance if, and only if, (n_{12},n_{21}) has a generator pair.

Definition 10. Let J_1,J_2,T_1,T_2 be elements of K, and (n_{12},n_{21}), $(\bar{n}_{12},\bar{n}_{21})$ strong Galois correspondances between J_1,J_2 and T_1,T_2 respectivelly. We say that (n_{12},n_{21}) i $(\bar{n}_{12},\bar{n}_{21})$ are equivalent if there exist two increasing one-one mappings f from J_1 to T_1, and g from J_2 to T_2, such that $g\circ n_{12}=\bar{n}_{12}\circ f$.

Next properties are deduce immediately from the definition.

1 - All strong Galois correspondances between J_1 and J_2 are equivalents. In fact, if (n_{12},n_{21}) and $(\bar{n}_{12},\bar{n}_{21})$ are two strong Galois correspondances between J_1 and J_2 by taking $f = j|_{J_1}$ and $g = \bar{n}_{12}\circ n_{21}$.

2 - If then exist an increasing one-one mapping f from J_1 to T_1, any strong Galois correspondence between J_1 and J_2 is equivalent to any strong Galois correspondence between T_1 and T_2.

In fact, let (n_{12},n_{21}) and $(\bar{n}_{12},\bar{n}_{21})$ be two strong Galois correspondences between J_1, J_2 and T_1,T_2 respectivelly. If we define $g=\bar{n}_{12}\circ f\circ n_{21}$ it is evident that $g\circ n_{12}=\bar{n}_{12}\circ f$.

II.2. <u>Characterisation of Isomorphisms.</u>

Let $R=\{J_x,\ x\in X\}$ and $R'=\{J'_x,\ x\in X\}$ be two families of elements of K, in all this part we consider the sublattices of $L(X)$, $P_R(X)$ and $P_{R'}(X)$.

Given, two De Morgan algebras $(P_R(X),\cap,\cup,n)$ and $(P_{R'}(X),\cap,\cup,\bar{n})$, the problem is reduced to find which isomoprphisms between the lattices $(P_R(X),\cap,\cup)$ and $(P_{R'}(X),\cap,\cup)$ are morphisms with respect to negations n and $\bar{n}$.

<u>Theorem 5.</u> Let n and $\bar{n}$ be two strong negations of $P_R(X)$ and $P_{R'}(X)$ respectivelly that fulfil the E.P. Let $\{n_x, x\in X\}$ and $\{n'_x, x\in X\}$ be the families of strong negation functions that define n and $\bar{n}$.

$(P_R(X),n)$ is isomorphic to $(P_{R'}(X),\bar{n})$ if, and only if, there exist σ, permutation of X, such that for any $x\in X$, we have that:

i) There exist an increasing one-one mapping between J_x and $J_{\sigma(x)}$.

ii) n_x and $n'_{\sigma(x)}$ have the same number of fixed points.

Proof. Given an isomorphism H from $(P_R(X),n)$ to $(P_{R'}(X),\bar{n})$, from Theorem 2 we know that there exist σ, permutation of X and a family of increasing one-one mappings $\{f_x : J_x \rightarrow J'_{\sigma(x)},\ x\in X\}$ such that $H(A)=\bigcup_{x\in X} \delta_{\sigma(x)}^{f_x(A(x))}$.

Otherwise for all $x\in X$ and $\alpha\in J_x$, $H(n(\delta_x^{\alpha}))=\bar{n}(H(\delta_x^{\alpha}))$, which is equivalent to $\delta_{\sigma(x)}^{f_x(n_x(\alpha))} = \delta_{\sigma(x)}^{\bar{n}_{\sigma(x)}(f_x(\alpha))}$, from where $f_x\circ n_x=\bar{n}_{\sigma(x)}\circ f_x$, that implies (Proposition 6) that n_x and $\bar{n}_{\sigma(x)}$ have the same number of fixed points.

Reciprocally, if the condition of theorem is fulfiled then proposition 6 say that, for all x of X, we may choose f_x, increasing one-one mapping from J_x to $J'_{\sigma(x)}$, such that $f_x \circ n_x = n'_{\sigma(x)} \circ f_x$ (1).

We define, for all A of $P_R(X)$, $H(A) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}$. Because of Theorem 2 we know that H is an isomorphism between the lattices $P_R(X)$ and $P_{R'}(X)$. Then, for all A of $P_R(X)$, we have

$$H(n(A)) = H(\bigcap_{x \in X} n(\delta_x^{A(x)})) = \bigcap_{x \in X} H(\bar{\delta}_x^{n_x(A(x))})$$

$$= \bigcap_{x \in X} \bar{\delta}_{\sigma(x)}^{f_n(n_x(A(x)))}$$

$$= \bigcap_{x \in X} \bar{\delta}_{\sigma(x)}^{n'_{\sigma(x)}(f_x(A(x))} \qquad \text{(for (1))}$$

$$= \bar{n}(\bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}) = \bar{n}(H(A)) \ ■$$

If $S = \{S_x, x \in X\}$ is a family of elements of K, symmetrical respect to 1/2 ($S_x = 1 - S_x$) and N is the strong negation of $P_S(X)$ p.f.e. by the family of strong negation functions $\{1-j: S_x \to S_x,\ x \in X\}$, then:

<u>Corollary 3.</u> All De Morgan algebra defined in a R-maximal lattice from a strong negation which fulfils the E.P. is isomorphic to an algebra of the kind $(P_S(X), N)$.

Proof. We consider $(P_R(X), n)$ a De Morgan algebra where n is the negation p.f.e. by the family of strong negation functions $\{n_x: J_x \to J_x,\ x \in X\}$.

For every $x \in X$ let f_x be an additive generator of n_x, hence $n_x = f_x^{-1} \circ (1-j) \circ f_x$ and $f_x(J_x) = 1 - f_x(J_x)$. We write $S_x = f_x(J_x)$, and $(P_R(X), n)$ is isomorph to $(P_S(X), N)$, because if we take $\sigma = j$ it is immediate that conditions i) and ii) of previous Theorem are fulfiled. ■

Theorem 6. We consider n and $\bar{n}$ two strong negations of $P_R(X)$ and $P_{R'}(X)$ respectivally, that fulfil the G.E.P. Let $\{(n_{xr(x)}, n_{r(x)x}); x \in X\}$ and $\{(\bar{n}_{xs(x)}, \bar{n}_{s(x)x}), x \in X\}$ to the families of strong Galois correspondences which define at n and $\bar{n}$.

$(P_R(X),n)$ is isomorph at $(P_{R'}(X),\bar{n})$ if, and only if, there exist σ, permutation of X, that fulfils:

i) $\sigma \circ r = s \circ \sigma$.

ii) For all $x \in X$, there exist an increasing one-one mapping between J_x and $J'_{\sigma(x)}$.

Proof. If H is an isomorphism from $(P_R(X),n)$ to $(P_{R'}(X),\bar{n})$, we know for theorem 2 that $H(A) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}$ for all A of $P_R(X)$, where σ is a permutation of X and f_x is an increasing one-one mapping from J_x to $J'_{\sigma(x)}$, for all $x \in X$. As H is an isomorphism of algebras we have that for all x of X, $H(n(\delta_x)) = \bar{n}(H(\delta_x))$, which is equivalent to $\delta_{\sigma(r(x))} = \delta_{s(\sigma(x))}$ hence we obtain $\sigma \circ r = s \circ \sigma$.

Reciprocally, if condition (i) is fulfiled then any strong Galois correspondence between J_x and $J_{r(x)}$ is equivalent to any strong Galois correspondence between $J'_{\sigma(x)}$ and $J'_{s(\sigma(x))}$. So, for any pair we know that there exist two one-one mappings $f_x, f_{r(x)}$ from J_x to $J'_{\sigma(x)}$ and from $J_{r(x)}$ to $J'_{s(\sigma(x))}$ respectively, such that

$$f_{r(x)} \circ n_{xr(x)} = \bar{n}_{\sigma(x)s(\sigma(x))} \circ f_x \quad (1).$$

We define H, isomorphism from the lattice $P_R(X)$ to the lattice $P_{R'}(X)$ in the following form:

$$H(A) = \bigcup_{x \in X} \delta_{\sigma(x)}^{f_x(A(x))}, \text{ for all } A \in P_R(X).$$

To see that H is morphism with respect to the negations, it is only necessary to prove that $H(n(\delta_x^\alpha)) = \bar{n}(H(\delta_x^\alpha))$ for all $x \in X$

and all $\alpha \in J_x$.

$$H(n(\delta_x^\alpha)) = H(\bar{\delta}_{r(x)}^{n_{xr(x)}(\alpha)}) = \bar{\delta}_{\sigma(r(x))}^{f_{r(x)}(n_{xr(x)}(\alpha))} \qquad \text{(for i), (1))}$$

$$= \bar{\delta}_{s(\sigma(x))}^{\bar{n}_{\sigma(x)s(\sigma(x))}(f_x(\alpha))}$$

$$= \bar{n}\,(\delta_{\sigma(x)}^{f_x(\alpha)}) = \bar{n}(H(\delta_x^\alpha)) \ ■$$

If $S=\{S_x, x \in X\}$ is a family of elements of K with the property that there exists t, permutation of X such that $t\circ t=j$, such that $S_{t(x)} = 1-S_x$, for all x of X; and N the negation of $P_S(X)$ generated by the family of Galois correspondences $\{(1-j,1-j):S_x \to S_{t(x)};\ x \in X\}$, then:

<u>Corollary 4.</u> Any De Morgan algebra defined in a R-maximal lattice from a negation which fulfil the G.E.P. is isomorphic to an algebra of the kind $(P_S(X),N)$.

Proof. We consider an De Morgan algebra $(P_R(X),n)$ where n is the negation generated by the family of strong Galois correspondences $\{(n_{xs(x)}, n_{s(x)x}),\ x \in X\}$. For any Galois correspondence of the family we choose a generator pait $(f_x, f_{s(x)})$. That means $n_{xs(x)}=f_{s(x)}^{-1} \circ (1-j)\circ f_x$, $n_{s(x)x}=f_x^{-1} \circ (1-j) \circ f_{s(x)}$ and $f_x(J_x)=1-f_{s(x)}(J_{s(x)})$. If we define, for all x of X, $S_x=f_x(J_x)$ and t=s, and if we take $\sigma=j$, it is clear that $(P_R(X),n)$ and $(P_S(X),N)$ are isomorphs, since it is immediate to see that the conditions i) and ii) of theorem 6 are fulfiled. ■

III. Classes of De Morgan Algebras of Fuzzy Sets.

In this part our aim is to study classes of De Morgan algebras of fuzzy sets with respect to isomorphisms. We could have considered first classes of R-maximals lattices, but Theorem 2 says that these classes only depend on the respective families R of possible values. It is for that reason that here we treat directely the problem of classification of algebras where negation which gives the algebra will play a basic role.

Proposition 8. If two lattices of fuzzy sets are isomorphics any negation in one of them defines a negation in the other, such that the corresponding De Morgan algebras are equivalents.

Proof. Let H be an isomorphism of lattices from $(P_R(X),\cap,\cup)$ to $(P_{R'}(X),\cap,\cup)$ an n a strong negation of $P_R(X)$. It is immediatly proved that $\bar{n}=H\circ n\circ H^{-1}$ is a strong negation of $P_{R'}(X)$ such that H is an isomophism between $(P_R(X),n)$ and $(P_{R'}(X),\bar{n})$ ■

The previous proposition says that fixed a family R in all classes of algebras defined in lattices R isomorphic to $(P_R(X),\cap,\cup)$ there is an algebra of the kind $(P_R(X),n)$. Then we are enabled to find the classes of algebras defined in lattices isomorphics by fixing a family R and by studying the classes of algebras of the kind $(P_R(X),n)$.

On the other hand, any isomorphism between algebras transform Boole algebras in Boole algebras and we have seen that $H|_{P(X)}$ it has be an automorphism of lattice P(X). Then, it is immediatly proved that if n and $\bar{n}$ are two negations of $P_R(X)$ that fulfil the E.P. and the E.G.P. respectivally, $(P_R(X),n)$ and $(P_R(X),\bar{n})$ can not be equivalent since the first contain the classical sets as Boole subalgebra and the second does not. So we will divide the study of the classes of algebras by distinguishing two cases.

a) Classes of De Morgan algebras $(P_R(X),n)$ where n fulfils the E.P.

Theorem 5 and the Corollary 3 say that

a_1) If $P_R(X)=\underset{\sim}{P}(X)$, any De Morgan Algebra is isomorphic to $(\underset{\sim}{P}(X),N)$, where N is defined by $[N(A)](x)=1-A(x)$. We have, then, only one classe of equivalence that will have as "canonic" element $(\underset{\sim}{P}(X),N)$.

a_2) If $P_R(X) = P_J(X)$, then we will have as classes of equivalence $(P_J(X),n)$ as possibles partitions of X_1 no-equipotents to X could be considered in two subsets $X_1, \bar{X}_1$; where $X_1=\{x \in X,\ n_x$ has a fixed point$\}$ and $\bar{X}_1=\{x \in X,\ n_x$ has not fixed point$\}$. Every class of equivalence has "canonic" elements of the kind $(P_S(X),\ N|_S)$ where:

i) S is a family S_x; $x \in X$ of sets symmetrics respect to 1/2 such that $1/2 \in S_x$ for $x \in X_1$ and $1/2 \notin S_x$ for $x \in \bar{X}_1$.

ii) $N|_S$ is the negation defined by

$$[(N|_S)(A)](x) = 1-A(x)$$

We remark that if in J we can only define negations with the same number of fixed points, then we have only one class of algebras.

a_3) In the general case in order to have $(P_R(X),n)$ and $P_R(X),n')$ isomorphics, it is necessary the existence of a permutation σ of X such that n_x and $n'_{\sigma(x)}$ have the same number of fixed points, and the existence of an increasing one-one mapping between J_x and $J_{\sigma(x)}$. In this case, every class will have "canonic" elements of kind the $(P_S(X),\ N|_S)$ where:

i) $S=\{S_x \in K;\ x \in X\}$ is a family of symmetry sets with respect to 1/2, such that $1/2 \in S_x$ if, and only if,

n_x has fixed point.

ii) $N|_S$ is the negation defined by $[(N|_S)(A)](x)=1-A(x)$.

b) <u>Classes of De Morgan algebras $(P_R(X),n)$, where n fulfils the G.E.P.</u>

In order to give the classification, a part from results of Theorem 6 and Corollary 4, we need remember that the <u>conjugation class of a permutation t of X</u> is the set $\{\sigma\circ t\circ\sigma^{-1}$, σ is a permutation of X$\}$.

b_1) If $P_R(X)=\underset{\sim}{P}(X)$, there are as equivalence classes as conjugation classes of involutives permutations of X. Every class will has "canonic" elements of the kind $(\underset{\sim}{P}(X),H_t\circ N)$, where t is a permutation of X and $H_t\circ N$ is the negation defined by

$$[(H_t\circ N)(A)](x) = 1-A(t(x)).$$

b_2) If $P_R(X)=P_J(X)$, then for any class of conjugations of involutive permutations of X there are as equivalence classes as partitions of X in two subsets no-equipotents $X_1,\bar{X}_1$ defined like in a_2). Every class will has "canonic" elements of the kind $(P_S(X),H_t\circ N)$, where S is a family $\{S_x|x\in X\}$ with the same conditions that in a_2) and $H_t\circ N$ is the negation defined by $[(H_t\circ N)(A)](x) = 1-A(t(x))$.
If J only admit negations with same number of fixed points then the equivalence classes coincide with the classes of involutive permutations of X.

b_3) In the general case, classes depent on both the conjugations classes of involutive permutations and on the possible permutations σ of X in the way that for any $x\in X$, J_x and $J_{\sigma(x)}$ are increasintly bijectable, such a combination is dificult to express together, that is why we believe that the best explanation is the result already given in Corolary 4.

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