STOCHASTICA Vol. VIII, Nº 3, 1984

# ON MEASURES OF CONCORDANCE*

Marco Scarsini

*ABSTRACT*

*We give a general definition of concordance and a set of axioms for measures of concordance. We then consider a family of measures satisfying these axioms. We compare our results with known results, in the discrete case.*

## 0. Introduction.

The intuitive idea that underlies the concept of concordance is this: Two random variables (r.v.'s) X and Y are concordant when large values of X go with large values of Y. Some attempts have been made to formulate this concept precisely, but only under the condition that the joint distribution fuction (d.f) has fixed marginals (Tchen (1980), Consonni and Scarsini (1982)). A general definition of concordance is given here for r.v.'s with continuous marginals, using the concept of a copula (Sklar (1959) This definition is then generalized to any r.v.'s. The definition proves to be consistent with the idea that concordance is invariant with respect to monotone increasing transformations of the random variables.

* Partially supported by GNAFA-CNR.
KEYWORDS: Concordance, copula. Fréchet class.
AMS 1970 subject classification: 62H99.

In addition the problem of finding some measures of concordance consistent with the definition just given is examined. Some axioms for measures of concordance are proposed, and a class of measures satisfying them is provided. Some of the usual measures of concordance such as Spearman's $\rho$ and Kendall's $\tau$ are examined in detail. In particular, in the case of discrete distributions, their usual expressions, and the ones obtained in terms of copulas are compared.

## 1. Definition of concordance.

Consider the d.f.'s $H(x,y)$ having fixed marginals $F(x)$ and $G(y)$, viz. the d.f.'s belonging to the Fréchet class $\Gamma(F,G)$ (Fréchet (1951)). If a point $(x^0,y^0)\varepsilon R^2$ is fixed, it is possible to determine four subsets of $\mathbb{R}^2$:

$$Q_1(x^0,y^0) = \{(x,y)\varepsilon\, \mathbb{R}^2 : x \leqslant x^0, y \leqslant y^0\}$$

$$Q_2(x^0,y^0) = \{(x,y)\varepsilon\, \mathbb{R}^2 : x \leqslant x^0, y > y^0\}$$

$$Q_3(x^0,y^0) = \{(x,y)\varepsilon\, \mathbb{R}^2 : x > x^0, y > y^0\}$$

$$Q_4(x^0,y^0) = \{(x,y)\varepsilon\, \mathbb{R}^2 : x > x^0, y \leqslant y^0\}.$$

On the quadrant $Q_1$ ($Q_3$) small (large) values of $X$ go with small (large) values of $Y$. On the quadrant $Q_2$ ($Q_4$) small (large) values of $X$ go with large (small) values of $Y$.

Definition 1. (Consonni and Scarsini (1982)). Let $H$, $H'\varepsilon\,\Gamma(F,G)$. $H$ is more concordant than $H'$ if, for all $(x,y)\varepsilon\, \mathbb{R}^2$,

$$\Pr\{(X,Y)\ \varepsilon[Q_1(x,y) \cup Q_3(x,y)]\,|H\}$$

$$\geqslant \Pr\{(X,Y)\varepsilon\ [Q_1(x,y) \cup Q_3(x,y)]\,|H'\}.$$

In this definition (and in the following ones, as well) "more concordant" is to be interpreted as an abbreviation of "not less concordant".

The following result is immediate:

<u>Proposition 1.</u> <u>Let</u> H, $H'\varepsilon\,\Gamma(F,G)$. H <u>is more concordant than</u> H' <u>if and only if</u> $H(x,y) \geqslant H'(x,y)$ <u>for all</u> $(x,y)\varepsilon\,\mathbb{R}^2$.

Tchen (1980) gave the following definition of concordance for distributions that concentrate their mass on finitely many atoms: "H is more concordant than H' if H can be obtained from H' by a finite number of repairings which add mass $\varepsilon$ at $(x,y)$ and $(x',y')$, while subtracting mass $\varepsilon$ at $(x',y)$ and $(x,y')$, where $x'>x$ and $y'>y$". Tchen's definition is consistent with Definition 1 for discrete distributions with finitely many atoms. It is relevant that, according to Definition 1, concordance is a matter of stochastic dominance. This is true only for d.f.'s belonging to the same Fréchet class, as the following counter-example shows: Let H be the d.f. which distributes its mass uniformly on the unit square $[0,1]^2$, and H' the d.f. which distributes its mass uniformly on the square $[1,2]^2$. Then H stochastically dominates H' (i.e., $H(x,y) \geqslant H'(x,y)$ for all $(x,y)\varepsilon\,R^2$) but it is not reasonable to regard H as more concordant than H' since each is obtained from the other by means of a translation along their common diagonal.

It is desirable to have a definition of concordance which regards situations such as the one in the above counter-example as equivalent. More precisely, it is desirable to have a definition according to which H and H' are equivalent if each may be obtained form the other by strictly increasing monotone transformations of the r.v.'s. For this purpose the concept of copula will be extremely useful (Sklar (1959), Schweizer and Sklar (1983), Schweizer and Wolff (1976), (1981)).

<u>Definition 2.</u> <u>A function</u> $\phi: R^2 \to \mathbb{R}$ <u>is called quasi-monotone if</u>

$$\phi(x,y) - \phi(x',y) - \phi(x,y') + \phi(x',y') \geqslant 0 \text{ whenever } x \geqslant x';\ y \geqslant y'.$$

We note that a function $\phi$ is quasi-monotone if and only if $\exp\{\phi\}$ is totally positive of order 2 (see Karlin (1968)).

<u>Definition 3. A (two-dimensional) copula is a map</u> $C:[0,1]^2 \to [0,1]$ <u>such that</u>

1) $C(u,0) = C(0,u) = 0$ <u>and</u> $C(u,1) = C(1,u) = u$, <u>for all</u> $u \in [0,1]$.
2) C <u>is quasi-monotone</u>.

We list some important properties of the copula C:

a) C is continuous.
b) $C^-(u,v) \triangleq \max(0,u + v - 1) \leqslant C(u,v) \leqslant C^+(u,v) \triangleq \min(u,v)$, for all $u, v \in [0,1]$.
c) If $(X,Y)$ is a pair of r,v,'s with d.f. $H \in \Gamma(F,G)$, then there exists a copula $C_{XY}$ such that:

$$H(x,y) = C_{XY}(F(x),G(y)), \text{ for all } x, y \in \mathbb{R}.$$

If F and G are continuous, then $C_{XY}$ is unique. Otherwise it is uniquely determined on Ran(F) x Ran(G), where Ran(f) denotes the range of the function f.

d) If f and g are strictly increasing on Ran(X), Ran (Y), respectively, then $C_{f(X)g(Y)} = C_{XY}$.
e) If f is strictly decreasing on Ran(X), then $C_{f(X)Y}(u,v) = v - C_{XY}(1 - u,v)$.
f) If F and G are continuous, then $C_{XY}$ is the restriction to the unit square of the joint d.f. of the r.v.'s F(X) and G(Y).
g) If f is strictly increasing (decreasing) on Ran(X), and Y=f(X), Then $C_{XY} = C^+$ $(C^-)$.

The copula represents the association between two r.v.'s, eliminating the influence of the marginals and hence of any monotone transformation on the marginals. It is therefore possible to reformulate concordance as a principle of stochastic dominance, with respect to copulas and no joint d.f.'s.

Definition 4. Let (X,Y) and (W,Z) be two pairs of r.v.'s with d.f.'s H and H', respectively (H and H' continuous). Then H is more concordant than H' if $C_{XY}(u,v) \geqslant C_{WZ}(u,v)$, for all $u,v \in [0,1]$.

In the following the expressions "(X,Y) is more concordant than (W,Z)", "$C_{XY}$ is more concordant than $C_{WZ}$", "H is more concordant than H'" will be used indifferently. The symbol $\geqslant$ will mean "more concordant than".

Definition 4 is consistent with Definition 1 when H and H' belong to the same Fréchet class.

A still more general definition, valid also for discontinuous d.f.'s, will be provided in Section 3.

## 2. Measures of concordance.

Consider the space $\mathcal{H}$ of joint d.f's with continuous marginals. The relation $\geqslant$ introduced in Definition 4 is a partial order in $\mathcal{H}$. The problem of measuring concordance may be viewed as that of establishing a total order on $\mathcal{H}$ consistent with the partial order $\geqslant$. This total order may be constructed by means of a map $\mathcal{J}:\mathcal{H} \to A$, where A is a totally ordered set (usually an interval in R). This map $\mathcal{J}$ is called a measure of concordance if it satisfies the following axioms (most of them are self-evident). Let (X,Y) be distributed according to the d.f. H (indicate this as $(X,Y) \sim H$), and define $I(X,Y)$ as $\mathcal{J}(H)$.

1. Domain: $I(X,Y)$ is defined for any (X,Y) with continuous d.f.
2. Symmetry: $I(X,Y) = I(Y,X)$.
3. Coherence: $I(X,Y)$ is monotone in $C_{XY}$, i.e., if $C_{XY} \geqslant C_{WZ}$, then $I(X,Y) \geqslant I(W,Z)$.
4. Range: $-1 \leqslant I(X,Y) \leqslant 1$.
5. Independence: $I(X,Y)=0$ if X and Y are stochastically independent.

6. Change of sign: $I(-X,Y) = -I(X,Y)$.
7. Continuity: If $(X,Y) \sim H$ and $(X_n,Y_n) \sim H_n$ $(n \in \mathbb{N})$, and if $H_n$ converges pointwise to $H$ ($H_n$ and $H$ continuous), then $\lim_{n\to\infty} I(X_n,Y_n) = I(X,Y)$.

Theorem 1. Let $I(X,Y)$ satisfy axioms 1-7. Then

a) if $f$ and $g$ are both strictly increasing or decreasing on $\mathrm{Ran}(X)$, $\mathrm{Ran}(Y)$, then $I(f(X),g(Y)) = I(X,Y)$.

b) $I(X,Y) = 1$ $(-1)$ if $Y = f(X)$ with $f$ a.s. strictly increasing (decreasing) on $\mathrm{Ran}(X)$.

Proof. a) Let $W = f(X)$ and $Z = g(Y)$ with $f$ and $g$ both strictly increasing on $\mathrm{Ran}(X)$, $\mathrm{Ran}(Y)$. Then $C_{XY} = C_{WZ}$ i.e., $C_{XY} \geqslant C_{WZ}$ and $C_{WZ} \geqslant C_{XY}$. By Axiom 3. $I(X,Y) \geqslant I(W,Z)$ and $I(X,Y) \leqslant I(W,Z)$, i.e., $I(X,Y) = I(W,Z)$. Now let $f$ and $g$ be decreasing, whence $-f$ and $-g$ are increasing. By Axiom 6. (applied twice) $I(-f(X),-g(Y)) = I(F(X),g(Y)) = I(X,Y)$ (the second equality given by a)).

b) If $f$ is increasing (decreasing), then $C_{Xf(X)}=C^+$ $(C^-)$. By 3 and 4, $I(C^+) = 1$, $I(C^-) = -1$.

The problem is now to find a class of measures that satisfy 1-7. We shall make use of some results concerning quasi-monotone functions.

Theorem 2. (Cambanis, Simons and Stout (1976), Tchen (1980)). Let $\phi$ be quasi-monotone, and $H$, $H' \in \Gamma(F,G)$ such that $H \geqslant H'$. Then $\int_{\mathbb{R}^2}\phi\, dH \geqslant \int_{R^2}\phi\, dH'$ provided the integrals exist. Conversely, if $\int_{\mathbb{R}^2}\phi\, dH \geqslant \int_{R^2}\phi\, dH'$ for any $H$, $H'$ such that $H \geqslant H'$, then $\phi$ is quasi-monotone.

Theorem 3. Let $I(X,Y)$ satisfy axioms 1 -7. If the joint d.f. of $(X,Y)$ is normal, with correlation coefficient $r$, then $I(X,Y)$ is an increasing function of $r$.

Proof. Let H be bivariate normal. Assume, without any loss of generality, that H is standard. Then $r = \int_{\mathbb{R}^2} xy\, dH(x,y)$ is monotone in H and is a one-to-one function from the space of standard bivariate normal d.f.'s to $[-1,1]$. Since $r=\int_0^1\int_0^1 N^{-1}(u)N^{-1}(v)dC(u,v)$, where N is the standard univariate normal, then r is also a monotone one-to-one function from the space of copulas of normal distributions to $[-1,1]$. Thus $I(X,Y)$ is monotone in $C_{XY}$ and hence monotone in r.

The following theorem describes a class of measure of concordance:

Theorem 4. Let $\psi$ be a bounded monotone odd function defined on $[-\frac{1}{2},\frac{1}{2}]$. Then $I(X,Y)$ satisfies axioms 1-7 if

$$I(X,Y) = \int_0^1\int_0^1 k\, \psi(u-\tfrac{1}{2})\psi(v-\tfrac{1}{2})\, dC_{XY}(u,v) \tag{1}$$

with $k = (\int_0^1 \psi^2(u-\frac{1}{2})\, du)^{-1}$.

Proof. 1) Since $\psi$ is bounded, the integral in (1) exists for any copula $C_{XY}$.

2) Evident, since $C_{XY}(u,v) = C_{YX}(v,u)$.

3) The function $\phi(u,v) \triangleq \psi(u-\frac{1}{2})\psi(v-\frac{1}{2})$ is quasi monotone, since $\psi$ is monotone; so Theorem 1 applies.

4) $I(X,Y)$ is included between two extremes, attained at $C^-$ and $C^+$ (-1 and 1, respectively). We recall that $C^+$ concentrates the mass on the diagonal $u = v$, and $C^-$ concentrates the mass on the diagonal $u = 1 - v$.

5) If X and Y are stochastically independent, then $C_{XY}(u,v) = uv$, so $\int_0^1\int_0^1 k\, \psi(u-\frac{1}{2})\psi(v-\frac{1}{2})\, du\, dv = k\, (\int_0^1\psi(u-\frac{1}{2})du)^2=0$, since $\psi$ is odd.

6) $C_{-XY}(u,v) = v - C_{XY}(1-u,v)$. Hence

$$\begin{aligned} I(-X,Y) &= \int_0^1\int_0^1 k\ \psi(u-\tfrac{1}{2})d(v - C_{XY}(1-u,v)) \\ &= -\int_0^1\int_0^1 k\ \psi(u-\tfrac{1}{2})\psi(v-\tfrac{1}{2})\ dC_{XY}(1-u,v) \\ &= -\int_0^1\int_0^1 k\ \psi(-z+\tfrac{1}{2})\psi(v-\tfrac{1}{2})\ dC_{XY}(z,v) \\ &= -I(X,Y). \end{aligned}$$

7) $I(X_n,Y_n) = \int_0^1\int_0^1 k\ \psi(u-\frac{1}{2})\psi(v-\frac{1}{2})\ dC_{X_nY_n}(u,v)$.

If $H_n \to H$ pointwise (with $H$ continuous), then $C_{X_nY_n} \to C_{XY}$ pointwise. The theorem of Helly-Bray gives the desired continuity result:

$$I(X_n,Y_n) \to I(X,Y).$$

The class of measures (1) does not exhaust the family of measures of concordance. It contains measures such as Blomquist q (choose $\psi(x) = \mathrm{sgn}(x)$), and Spearman's $\rho$ ($\psi$ = identity function)

$$\rho = 12 \int_0^1\int_0^1 (u-\tfrac{1}{2})(v-\tfrac{1}{2})\ dC(u,v).$$

This expression is equivalent to the following (Schweizer and Wolff (1981))

$$\rho = 12 \int_0^1\int_0^1 (C(u,v) - uv)\ du\ dv.$$

Equivalence may be proved either by integrating by parts the bivariate integral in any of the two formulae (see Picone and Viola (1952) for the integration by parts of bivariate integrals), or by using a lemma of Hoeffding (see Lehmann (1966)). The class (1) contains neither Kendall's $\tau$ nor Gini's cograduation coefficient G. These two measures also satisfy axioms 1-7, as is shown by

<u>Theorem 5.</u> a) <u>Kendall's</u> $\tau$, <u>which is given by</u>

$$\tau = 4 \int_0^1\int_0^1 C_{XY}(u,v)\ dC_{XY}(u,v) - 1 \tag{2}$$

satisfies axioms 1-7.

b) Gini's G, which is given by

$$G = 2 \int_0^1\int_0^1 (|1 - u - v| - |u - v|) dC_{XY}(u,v) \tag{3}$$

satisfies axioms 1-7.

Proof. a) Except for 3 and 7, all of the properties either are trivial, have already been proved by Kruskal (1958), or may be proved by considerations similar to those given in the proof of Theorem 2. Thus we need only prove 3 and 7. Here and in the following, integrals of the form $\int \phi dC$ are over $[0,1] \times [0,1]$.

3. Let $C_{XY} \geqslant C_{WZ}$. Then

$$\int C_{WZ}\ dC_{WZ} \leqslant \int C_{WZ} dC_{XY} \leqslant \int C_{XY} dC_{XY}$$

where first inequality holds by virtue of the quasi-monotonicity of $C_{WZ}$.

7. If $H_n \to H$ pointwise (with H continuous), then $C_{X_nY_n} \to C_{XY}$ pointwise. It is necessary to prove that $\int C_{X_nY_n} dC_{X_nY_n} \to \int C_{XY} dC_{XY}$. We use the following lemma (a more general form of which is proved in Serfozo (1982)).

Lemma 1. (Serfozo (1982), Theorem 3.5) Let $f, f_1, f_2, \ldots$ be nonnegative. If $\mu_n \to \mu$ weakly (where $\mu_n$, $\mu$ are probability measures on $X \subset R^k$), $f_n \to f$ continuously (i.e., $\lim_{n\to\infty} f_n(x_n) = f(x)$ for any $x_n \to x$, $x \in X$) and $\int f_n d\mu_n < \infty$, $n \geqslant 1$, then the following statements are equivalent:

i) $\int f_n d\mu_n \to \int f\ d\mu < \infty$.

ii) $f_n$ is uniformly $\{\mu_n\}$-integrable (i.e., $\lim_{a\to\infty} \sup_n \int_{|f_n|>a} |f_n| d\mu_n = 0$

If we choose $[0,1]^2$ as $X$, $C_{X_n Y_n}$ as $f_n$, and the measure induced on $[0,1]^2$ by $C_{X_n Y_n}$ as $\mu_n$, then the hypotheses of Lemma 2 are satisfied and ii) holds, since $C_{X_n Y_n}$ are uniformly bounded. Hence i) holds, as well, and

$$\int C_{X_n Y_n} \, dC_{X_n Y_n} \to \int C_{XY} \, dC_{XY}.$$

b) The proof of each of the properties is similar to that of Theorem 2, or can be achieved by straighforward calculations.

Before we end this section some comment is necessary about Axiom 5. Independence is sufficient for the vanishing of a measure of concordance but is not necessary. Imposing independence as a necessary and sufficient condition for a measure of concordance to the zero would be incompatible with any of the measures we have considered.

<u>Theorem 4.</u> <u>Let</u> $I(X,Y) = \int \phi \, dC_{XY} < \infty$ with $\phi$ <u>quasi-monotone. Then</u> $I(X,Y) = 0$ <u>does not imply that</u> X <u>and</u> Y <u>are stochastically independent.</u>

Proof. Let $C^\circ(u,v) = uv$ be the copula of $(X,Y)$ when they are stochastically independent. We then have:

$$I^- \triangleq \int \phi \; dC^- \leqslant \int \phi dC^o \leqslant \int \phi dC^+ \triangleq I^+.$$

Now suppose $\int \phi dC^o = 0$ and let $\varphi = \{C_\alpha : C_\alpha = \alpha C^- + (1-\alpha) C^+, \alpha \in [0,1]\}$. Clearly, such $C_\alpha \in \varphi$ is a copula and $C^o \notin \varphi$. Consider the function $\psi : [0,1] \to \mathbb{R}$ so defined:

$$\begin{aligned} \psi(\alpha) &= \int \phi dC_\alpha \\ &= \alpha \int \phi dC^- + (1-\alpha) \int \phi dC^+ \\ &= \alpha I^- + (1-\alpha) I^+. \end{aligned}$$

It is evident that $\psi$ is a continuous function, whose range is the interval $[I^-, I^+]$. Then there exists a copula $C_{\alpha*} \neq C^o$ such that $I_{\alpha*} \stackrel{\Delta}{=} \int \phi \, dC_{\alpha*} = 0$.

Rényi (1959) gave a set of axioms for measures of dependence. Schweizer and Wolff (1976), (1981) modified them in order to obtain axioms for monotone dependence.

Dependence is a matter of association of X and Y along any (measurable) function, i.e., the more X and Y tend to cluster around the graph of a function, either $y = f(x)$, or $x = g(y)$, the more they are dependent. Monotone dependence is a matter of association with respect to a (strictly) monotone function (indifferently increasing or decreasing). The minimum dependence, as well as the minimum of monotone dependence, corresponds to independence.

Concordance, on the other hand, takes into account the kind of monotonicity (whether increasing or decreasing), so that the maximum of concordance is attained when a strictly monotone increasing relation exists between the variables, and the minimum of concordance (viz. perfect discordance) is attained when a relation exists that is strictly monotone decreasing.

## 3. Discrete case.

A problem arises in extending the definitions and theorems given in the preceding sections to the case of discontinuous joint d.f.'s. Even the definition of concordance is no longer acceptable as stated. The source of trouble is the non-unicity of the copula in the general case. Definition 3 must therefore be replaced by a (necessarily) more complicated one. First of all we have to define a subcopula.

Definition 5. Let A and B be two subsets of [0,1], containing 0 and 1.

A subcopula is a function $C^*: A \times B \to [0,1]$ which satisfies the following conditions

i) $C^*$ is quasi monotone.

ii) $C^*(u,1) = u$ for all $u \in A$.
$C^*(1,v) = v$ for all $v \in B$.

Definition 6. Let $(X,Y)$ and $(W,Z)$ be two pairs of r.v.'s with d.f.'s $H$ and $H'$ respectively ($H \in \Gamma(F,G)$; $H' \in \Gamma(F',G')$). Let $C^*_{XY}$ and $C^*_{WZ}$ be the respective subcopulas defined on $\mathrm{Ran}(F) \times \mathrm{Ran}(G)$ and $\mathrm{Ran}(F') \times \mathrm{Ran}(G')$ respectively. Let $\varphi^-_{XY}$ be the family of copulas $C^-_{XY}$, such that, for any $C_{XY}$ which satisfies

$$C^-_{XY}(u,v) = C^*_{XY}(u,v) = C_{XY}(u,v) \text{ for all } (u,v) \in [\mathrm{Ran}(F) \times \mathrm{Ran}(G)]$$

the inequality

$$C^-_{XY}(u,v) \geq C_{XY}(u,v) \text{ for all } (u,v) \in [0,1]^2$$

is impossible.

Let $\varphi^+_{WZ}$ be the family of copulas $C^+_{WZ}$ such that, for any $C_{WZ}$ which satisfies

$$C^+_{WZ}(u,v) = C^*_{WZ}(u,v) = C_{WZ}(u,v) \text{ for all } (u,v) \in [\mathrm{Ran}(F') \times \mathrm{Ran}(G')]$$

the inequality

$$C^+_{WZ}(u,v) \leq C_{WZ}(u,v) \text{ for all } (u,v) \in [0,1]^2$$

is impossible.

If, for any $C^-_{XY} \in \varphi^-_{XY}$ and $C^+_{WZ} \in \varphi^+_{WZ}$

$$C^-_{XY}(u,v) \geq C^+_{WZ}(u,v)$$

for all $(u,v) \in \{[\mathrm{Ran}(F) \times \mathrm{Ran}(G)] \cup [\mathrm{Ran}(F') \times \mathrm{Ran}(G')]\}$, then

$(X,Y)$ <u>is said to be more concordant than</u> $(W,Z)$.

The meaning of the definition is this: whenever the comparison between subcopulas is possible, then this is considered; otherwise each subcopula is compared with the "worst" copula compatible with the other subcopula. Definition 4 and Definition 6 coincide when H and H' are continuous.

Definition 6 is cumbersome and concordance is not easily testable through it in practice. In fact, the classes $\varphi^{-}_{XY}$ and $\varphi^{+}_{WZ}$ contain more than one copula, for which no explicit formula is obtainable, in general.

Difficulties grow when measures of concordance are examined. Consider expression (1). If H is continuous, this may be written:

$$I(X,Y) = \iint_{R^2} k\psi(F(x)-\tfrac{1}{2})\psi(G(y)-\tfrac{1}{2})\,dH(x,y). \qquad (4)$$

When H is discontinuous, the integral in (4) cannot be considered in the Riemann-Stieltjies sense, because the integrand and the integrator in general have common one-sided discontinuity points. If it is intended in the Lebesgue-Stielthes sense, then its value depends on the definition of a d.f., e.,e., $H(x,y) = \Pr\{X \leqslant x; Y \leqslant y\}$ or $H(x,y) = \Pr\{X<x; Y<y\}$. Even the definition of the normalizing constant k becomes difficult. These unpleasant features are not overcome by using copulas and the measure in the form (1), because there exist many copulas compatible with a joint d.f. and these different copulas give different values of the measure I. In the following, a criterion for choosing one copula among the admissible ones is proposed. In this way the problem of computing the measure of concordance is solved through (1), or through any other expression of a measure of concordance which involves only copulas (e.g. (3) and (4)).

The procedure that we are going to illustrate is the one used in Schweizer and Sklar (1974) to prove that every subcopula can be extended to a copula.

The criterion can be described as follows: Let $H \in \Gamma(F,G)$.

Split the unit square $[0,1]^2$ into four regions:

$$A \triangleq \mathrm{Ran}(F) \times \mathrm{Ran}(G),$$

$$B \triangleq \mathrm{Ran}(F) \times \{[0,1]\setminus \mathrm{Ran}(G)\},$$

$$D \triangleq \{[0,1]\setminus \mathrm{Ran}(F)\} \times \mathrm{Ran}(G),$$

$$E \triangleq \{[0,1]\setminus \mathrm{Ran}(F)\} \times \{[0,1]\setminus \mathrm{Ran}(G)\}.$$

On A a subcopula is uniquely defined.

The region B is the (at most countable) union of intervals $[a,b] \times (c,d)$ $(a \leqslant b; c < d)$, such that $(u,c)$, $(u,d) \in A$ for all $u \in [a,b]$. On each of these intervals in B we define a subcopula in this way:

$$C(u,v) = \alpha C(u,c) + (1-\alpha)C(u,d),$$

where $v = \alpha c + (1-\alpha)d$ $(0 \leqslant \alpha \leqslant 1)$.

We use an analogous procedure for D (interchanging the coordinates).

To complete the procedure we need only define a subcopula on E. This can be done in a similar way, taking into account the subcopulas defined on A, B and D.

Combining the subcopulas defined on A, B, D and E we obtain a copula compatible with the original d.f. H. This copula (indicate it as $C^0_{XY}$) may be justified as follows: Consider the d.f. H; then spread the mass concentrated by H on vertical segments uniformly on the rectangles with base 1/n on the left of these segments: spread the mass on horizontal segments uniformly on the rectangles with height 1/n below these segments; spread the mass of atoms uniformly on the square with side 1/n south-west of the atoms; call the d.f. that corresponds to this new situation $H_n$. Each of the $H_n$ so obtained has continuous marginals and hence an unique copula $C_n$ (say). By letting n go to the infinity, $H_n \to H$ weakly, and we obtain the desired copula $C^0_{XY} = \lim C_n$.

In other words we approximate H (which has discontinuous marginals) with a converging sequence of d.f.'s $H_n$ which have continuous marginals. We consider the sequence of corresponding copulas $C_n$ and choose the limit of $C_n$ as the copula of H. The limit copula is -evidently- dependent on the sequence $H_n$, in that two different sequences $\{H_n\}$ and $\{H'_n\}$ both converging weakly to H may generate two different limit copulas and a converging sequence $\{H''_n\}$ might even have no limit copula (e.g., take $H''_{2n} = H_n$, $H''_{2n+1} = H'_n$).

Our criterion ensures the existence of the limit copula: the one described above.

When (X,Y) consists of n atoms $((x_i,y_i),\ i = 2,\dots,n)$ each with mass 1/n and $x_i \neq x_j$, $y_i \neq y_j$ $i \neq j$, i.e., there are no ties, formulae exist for computing some usual measures of concordance such as Kendall's $\tau$, Spearman's $\rho$, etc. (see Kruskal (1958)). We shall compare results obtained by use of these formulae with those obtained by use of the limit copula $C^0_{XY}$.

Kendall's $\tau$. The copula $C_{XY}$ is uniquely defined on $J_n \times J_n$ where $J_n = (0,1/n,\dots,1)$. Let $(i/n,\pi(i)/n)$ be the points of $J_n \times J_n$ that correspond to the n atoms of H. Obviously $\{\pi(1),\pi(2),\dots\pi(n)\}$ is a permutation of $\{1,2,\dots,n\}$. We say that $(x_i,y_j)$ and $(x_j,y_j)$ are concordant, if $(x_i - x_j)\cdot(y_i - y_j) > 0$. Define $\Omega \triangleq$ (# concordant $(x_i,y_i)$ $(x_j,y_j)$, $i < j$).

We have:

$$\int_0^1\int_0^1 C^0_{XY}\, dC^0_{XY} = \sum_{i=0}^{n-1}\Big[\int_{i/n}^{(i+1)/n}\int_{\pi(i)/n}^{(\pi(i)+1)/n}(C_{XY}(i/n,\pi(i)/n)$$

$$+ n(u - i/n)(v - \pi(i)/n))\, d(n(u - i/n)(v - \pi(i)/n))\Big]$$

$$= \sum_{i=0}^{n-1}\Big[\int_0^{1/n}\int_0^{1/n} n^2\, uv\, du\, dv\Big] + \sum_{i=0}^{n-1}\Big[\int_0^{1/n}\int_0^{1/n} C_{XY}(i/n,\pi(i)/n)n\, du\, dv\Big]$$

$$= 1/(4n) + (1/n^2)\Omega$$

$$\tau = 4\int\int C^0_{XY}\, dC^0_{XY} - 1$$

$$= 1/n + (4/n^2)\Omega - 1.$$

When calculated by the usual formula, $\tau$ (indicate it as $\tau_S$) is:

$$\tau_S = 4\Omega/(n(n - 1)) - 1.$$

Hence

$$\tau = \tau_S(n - 1)/n.$$

Spearman's $\rho$.

$$\rho = 12 \int_0^1\int_0^1 uv\, dC^0_{XY}(u,v) - 3$$

$$= 12 \sum_{i=1}^{n} [ \int_{(i-1)/n}^{i/n} \int_{(\pi(i)-1)/n}^{\pi(i)/n} uv/n \, du\, dv ] - 3$$

$$= 12 \sum_{i=1}^{n} (\frac{2i-1}{2n^2})(\frac{2\pi(i)-1}{2n^2}) - 3$$

$$= \frac{1}{n^3}[ 12 \sum_{i=1}^{n} i\pi(i) - 3n(n + 1)^2].$$

When calculated by the usual formula, $\rho$ (indicate it as $\rho_S$) is:

$$\rho_S = \frac{1}{n(n^2-1)}(12 \sum_{i=1}^{n} i\pi(i) - 3n(n + 1)^2).$$

Hence

$$\rho = \rho_S(n^2 - 1)/n^2.$$

In both cases the measures of concordance calculated by means of copulas differ from the usual ones by a factor $k(n)$, say (different in each case), such that $k(n) \to 1$, as $n \to \infty$.

The advantage of calculating a measure of concordance by means of copulas is that it is not necessary to know the number of observations n, but just the joint d.f..

When calculation of measures of concordance is preformed according to our criterion, atoms are considered as if they were square intervals, within which the two r.v.'s are independent. Therefore a measure of concordance of a discrete distribution may be less than one, even if the ranks of the x's and the ranks of the y's match perfectly. Since in such a case it is usually required to a measure of concordance to be one, then it is possible to "normalize" the measure of concordance by means of the factor k(n). The behavior of k(n) may be explained as follows: if there are n observations with no ties, then each arom of the d.f. has mass 1/n. As n increases the mass of each atom decrases and the range of the unnormalized measure of concordance enlarge, and the limit range is just [-1,1].

## References


[1] CAMBANIS, S., SIMONS, G. and STOUT, W. (1976). Inequalities for Ek(X,Y) when the marginal are fixed. Z. Wahrscheinlichkeitstheorie verw. Gebiete 36, 285-294.

[2] CONSONNI, G. and SCARSINI M. (1982). Lo studio della concordanza nel contesto della teoria della variabilità superficiale. Statistica 42, 69-77.

[3] FRÉCHET, M. M. (1950). Sur les tableaux de corrélation dont les marges sont donées. Ann. Univ. Lyon Sec. A 14, 53-77.

[4] KARLIN, S. (1968). Total Positivity Vol I, Stanford University Press. Standord.

[5] KRUSKAL, H. (1958). Ordinal measures of association. J. Amer Statist. Assoc. 53, 814-861.

[6] LEHMANN, E. (1966). Some concepts of dependence. Ann. Math. Statist. 37, 1137-1153.

[7] PICONE M. and VIOLA, T. (1952). Lezioni sulla teoria moderna dell' integrazione. Einaudi, Torins.

[8] RÉNYI, A. (1959). On measures of dependence. Acta Math. Acad. Sci. Hungar. 10, 441-451.

[9] SCHWEIZER, B. and SKLAR, A. (1974). Operations on distribution functions not derivable from operations on random variables. Studia Math. 52, 43-52.

[10] SCHWEIZER, B. and SKLAR, A. (1983). Probabilistic Metric Spaces. Elsevier Science Publishing Co., New York.

[11] SCHWEIZER, B. and Wolff, E. F. (1976). Sur une measure de dépendance pour les variables aléatories. C. R. Acad. Sc. Paris Sér. A 283, 659-661.

[12] SCHWEIZER, B. and WOLFF, E. F. (1981). On nonparametric measures of dependence for random variables. Ann. Statist. 9, 879-885.

[13] SERFOZO, R. (1982). Convergence of Lebesgue integrals with varying measures. Sankya Ser. A 44, 380-402.

[14] SKLAR, A. (1959). Fonctions de réparition à n dimensions et leurs marges. Inst. Statist. Univ. Paris Publ. 8, 229-231.

[15] TCHEN, A. (1980). Inequalities for distributions with given marginals. Ann. Probability 8, 814-827.

ACKNOWLEDGEMENTS. Helpful comments by Professors Castagnoli and Sklar are gratefully acknowledged. Particular thanks are due to a referee for his careful reading and criticism.

Istituto di Matematica Finanziaria,
Università di Parma, via Kennedy, 6
I-43100, Parma, Italy.
Now visiting the Dep. of Mathematics,
The Univ. of Arizona, Tucson,AZ 85721
U.S.A.