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A CHARACTERIZATION OF SHAPLEY INDEX OF POWER VIA AUTOMORPHISMS

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ABSTRACT

In the class of complete games, the Shapley index of power is the characteristic invariant of the group of automorphisms, for these are exactly the permutations of players preserving the index.

1. Introduction.

In a proper simple game, a primary measurement of the strength of each player is given by a preorder relation associated with the filter of winning coalitions.

The Shapley index of power is compatible (in a monotonic sense) with this preorder. On the other hand, the equivalence relation associated with the preorder is the symmetry, closely related to the group of automorphisms.

In this way, when the preorder is total (complete games) a characterization of Shapley index is reached as the basic invariant of the group of automorphisms inside the permutations group of the players.

Completeness condition (fullfilled by a large class of games,

e. g. weighted majority games) cannot be eliminated, as the two final examples show.

A (monotonic) simple game is a pair (X,F) where X is a finite set and F is a filter of order in $P(X)$, that is, a non-empty collection of subsets of X (called the winning coalitions) such that if $A \subseteq B$ and $A \in F$ then $B \in F$.

A simple game (X,F) is proper if F is connected, that is, if $A \cap B \neq \emptyset$ when $A,B \in F$. Only proper simple games will be considered in the sequel, and so the two qualifying will be omitted.

The basis F_o of the filter F is the collection of the minimal winning coalitions (with respect to the inclusion ordering). An element $x \in X$ is a dummy if $x \notin A$ for every $A \in F_o$. On the other hand, an element $x \in X$ is said to have a veto if $x \in A$ for every $A \in F_o$. The set of dummy (resp. veto) elements of X will be denoted by N (resp. V).

A homomorphism of games $f : (X,F) \to (X',F')$ is a map $f:X \to X'$ such that $f(A) \in F'$ when $A \in F$. The identity map and also the composition (when possible) of two homomorphisms are homomorphisms. f is called an isomorphism if there exists a homomorphism $g: (X',F') \to (X,F)$ such that $g \circ f = id_X$ and $f \circ g = id_{X'}$. This is equivalent to say that f is bijective and satisfies $f(A) \in F'$ if and only if $A \in F$.

Let (X,F) be a game and $n = |X|$ the number of players. We consider the set T of all arrangements of X, that is, bijective maps $s : [1,n] \to X$, where $[1,n] = \{1,2,\ldots,n\}$. The pivot of a given $s \in T$ is the value $s_i \in X$ defined by $i=\min\{j \in [1,n] : s([1,j]) \in F\}$, and the power index (of Shapley and Shubik) is the map $p:X \to \mathbb{R}$ given by $p(x)=|T(x)|/n!$ for each $x \in X$, where $T(x)=\{s \in T : x=\text{pivot of } s\}$.

2. Automorphisms and symmetry.

An automorphism of a game (X,F) is an isomorphism $f: (X,F) \to (X,F)$. Provided with the ordinary composition of maps as a law of composition, the set of automorphisms of a game (X,F) forms a group, denoted by Aut(X,F), which is a subgroup of S(X), the symmetric group on X.

A game (X,F) is called symmetric if its group is Aut(X,F)= =S(X). There are non-isomorphic symmetric games over the same X (e. g. all non-weighted majority games), and hence the group does not determine the filter F.

Proposition 2.1. The index [S(X):Aut(X,F)] gives the number of filters over X isomorphic to F.

Proof: For each $f \in S(X)$, the collection f(F) is a filter over X isomorphic to F. $f \to f(F)$ is a surjective map of S(X) into the set of such filters. Moreover, f(F) = g(F) if and only if $g^{-1} \circ f \in Aut(X,F)$, and this is the condition for f and g to belong to the same left coset modulo Aut(X,F). Then there exists a bijection between S(X)/Aut(X,F) and the set of filters over X isomorphic to F.

Let (X,F) be a game. We define the symmetry relation in X by setting

$$x \; R \; y \quad \text{if and only if} \quad t_{xy} \in Aut(X,F)$$

where t_{xy} is the transposition of x,y, and t_{xx} is taken to be the identity. It is trivially verified that R is an equivalence relation, because $t_{yx} = t_{xy}$ and, for x,y,z pairwise distinct, $t_{xz} = t_{yz} \circ t_{xy} \circ t_{yz}$. The equivalence classes will be called symmetry classes.

Since the group Aut(X,F) operates on X, we shall consider for each $x \in X$ its orbit $0_x = \{y \in X: y = f(x) \text{ for some } f \in Aut(X,F)\}$. The (distinct) orbits give a partition of X, each one being invariant under each automorphism of the game. We recall that when the group is transitive there is only one orbit.

By the preceding definitions it is obvious that every orbit is a disjoint union of symmetry classes. Especially, N and V (if non-empty) are at the time orbits and symmetry classes. Two elements $x, y \in X$ are said to be equivalent if xRy, that is, if t_{xy} is an automorphism. To distinguish this situation, we say that x,y are relatively equivalent if there is some automorphism f (not necessarily the pure transposition) such that $y=f(x)$. In this case, x and y roles are also interchangeable, but it requires a simultaneous transformation of the rest (or part) of the set of players.

Proposition 2.2. Power index is invariant under automorphisms; in other words, it is a constant function over each orbit.

Proof: Let $f \in Aut(X,F)$. The map $s \to fos$ is a bijection of T with itself. Given $x \in X$, $fos \in T(f(x))$ when $s \in T(x)$, and hence $p(x) \leqslant p(f(x))$. By applying the same argument to f^{-1} we obtain $p(x) \geqslant p(f(x))$ and so $p(x) = p(f(x))$.

Corollary. Let Aut(X,F) be transitive (for example if (X,F) is symmetric). Then:

a) p is constant: $p(x) = 1/n$ for all $x \in X$

b) $N = \emptyset$

c) all symmetry classes have the same number of elements; in particular, either $V = X$ (and then $F = \{X\}$) or $V=\emptyset$.

Proof: a) and b) are obvious.

c) Let C, C' be two symmetry classes, $x \in C$, $x' \in C'$. Let $f \in Aut(X,F)$ such that $x' = f(x)$. If $y \in C$ and $y' = f(y)$, the relation

$$t_{x'y'} = f \circ t_{xy} \circ f^{-1}$$

gives x'Ry', and then $f(C) \subseteq C'$. By applying the same argument to f^{-1} we get $C' \subseteq f(C)$, and hence $|C| = |C'|$.

3. Two lemmas.

Let (X,F) be a game. We define the <u>substitution relation</u> in X by setting

$$x \; S \; y \quad \text{if and only if} \quad t_{xy}(F_x) \subseteq F_y,$$

where $F_x = \{A \in F\colon x \in A\}$ for every $x \in X$. Reflexive and transitive properties (easily verified) lead S to be a preorder relation, whose associated equivalence relation is precisely R, since $t_{xy} \in \mathrm{Aut}(X,F)$ if and only if $t_{xy}(F_x)=F_y$. Thus, an ordering is induced by S in the quotient set X/R: its minimum (resp. maximum) is the class N (resp. V) if non-empty.

We will now show a general result of group theory to be used later.

Let X be a finite set and G a subgroup of $S(X)$. The symmetry classes of X with respect to G can be defined in a similar way as done for the case $G=\mathrm{Aut}(X,F)$. Each G-orbit is also a disjoint union of G-classes of symmetry. If $U \subseteq X$, the group $S(U)$ can be viewed as a subgroup of $S(X)$ via a natural monomorphism. If $U,V \subseteq X$ are disjoint, the direct product $S(U) \times S(V)$ can also be embedded in $S(X)$.

<u>Lema 3.1.</u> Let G be a subgroup of $S(X)$. The following conditions are equivalent:

i) G is generated by its transpositions.

ii) each G-orbit is a G-class of symmetry.

iii) there are pairwise disjoint subsets $U_1,U_2,\ldots,U_r$ of X such that $G = S(U_1) \times S(U_2) \times \ldots \times S(U_r)$.

Proof: i) implies ii). Let x,y be in the same orbit. There exist $f \in G$ such that $y = f(x)$. From being $f = t_1 \circ t_2 \circ \ldots \circ t_i$ a product of transpositions belonging to G, it follows:

$$x \; R \; t_i(x), \; t_i(x) \; R \; (t_{i-1} o t_i)(x), \ldots,$$

$$\ldots, \; (t_2 o t_3 o \ldots o t_i)(x) \; R \; (t_1 o t_2 o \ldots o t_i)(x) = y.$$

By the transitivity of R we obtain xRy, and hence x and y are in the same G-class of symmetry.

ii) implies iii). Let $U_1, U_2, \ldots, U_r$ be the G-orbits. Then, as explained before the lemma, we have $G \subseteq S(U_1) \times S(U_2) \times \ldots \times S(U_r)$. But $U_1, U_2, \ldots, U_r$ are also the G-classes of symmetry, hence $S(U_1) \times S(U_2) \times \ldots \times S(U_r) \subseteq G$ and the equality follows.

iii) implies i). Since G is a product of symmetric groups, it is generated by transpositions.

Lema 3.2. Let (X,F) be a game. If xSy but $y \not S x$, then $p(x) < p(y)$.

Proof: The map $s \to t_{xy} o s$ is a permutation of the set T of the arrangements of X. Since xSy, y is the pivot of $t_{xy} o s$ when x is the pivot of s, and then $p(x) \leqslant p(y)$. Since $y \not S x$, there exists $A \in F_o \cap F_y$ such that $t_{xy}(A) \notin F$, and so $y \in A$ but $x \notin A$. If $i = |A|$, there are $(i-1)!(n-i)!$ arrangements of X whose pivot is y and whose inverse image (obtained by composing with $t_{xy}^{-1} = t_{xy}$) has not x as its pivot. We can suppose $1 < i < n$ to eliminate trivial cases, and hence

$$p(x) < p(x) + \frac{(i-1)!(n-i)!}{n!} \leqslant p(y).$$

4. Complete games.

A game (X,F) is said to be complete if the preorder S is total. For instance, all weighted majority games are complete. In this section we shall prove that power index is the characteristic invariant of the group of automorphisms of a complete game, and that the group splits into a direct product of symmetric groups. Some consequences are derived from these results.

Proposition 4.1. The automorphisms of a complete game (X,F) are exactly the permutations of X which preserve the power index, that is,

$$\mathrm{Aut}(X,F) = \{f \in S(X) : p(f(x)) = p(x) \text{ for all } x \in X\}.$$

Moreover, if $X_1, X_2, \ldots, X_k$ are the symmetry classes of (X,F),

$$\mathrm{Aut}(X,F) = S(X_1) \times S(X_2) \times \ldots \times S(X_k).$$

Proof: Since power index is invariant under automorphisms by Prop. 2.2., it suffices to prove that every permutation of X preserving the power index is an automorphism. Let f be one such permutation and let $y = f(x)$ for each x. From $y \not R x$, we must have ySx or xSy but not both. By Lemma 3.2 there is then a contradiction with the invariance of power index under f. Thus, $xRf(x)$ for each $x \in X$, f leaves invariant every symmetry class and then f is an automorphism.

Now, the direct product of the groups of permutations of symmetry classes is a subgroup of $\mathrm{Aut}(X,F)$. But each $f \in \mathrm{Aut}(X,F)$ has been proved to split into a product of permutations of these classes. The direct product, then, coincides with the group of automorphisms.

Corollary 1. Let (X,F) be a complete game. Then:

a) each orbit of $\mathrm{Aut}(X,F)$ is a symmetry class, that is, there are no pairs of elements only relatively equivalent.

b) $p(x) = p(y)$ if and only if xRy.

c) the power index values over the symmetry classes (or orbits) are all distinct.

d) the number of filters over X isomorphic to F is the combinatorial number $C^n_{n_1, n_2, \ldots, n_k}$, where $n_i = |X_i|$ for $i = 1,2,\ldots,k$.

Proof: a) By the last proposition and Lema 3.1.

b) p(x) = p(y) implies that x,y are in the same orbit, that is, in the same class; the converse comes from Prop. 2.2.

c) By combining a) and b).

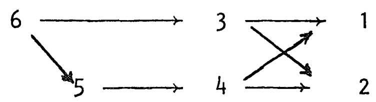
d) It is a consequence of Prop. 2.1, having in mind the decomposition of Aut(X,F) as a direct product.

Corollary 2. Let (X,F) be a complete game. If power index is constant over X or Aut(X,F) is transitive, then the game is symmetric.

Proof: By a) and c) of Corollary 1.

The following counterexamples show the need of completeness.

Counterexample 1. Let X = {1,2,3,4,5,6} and F the filter whose basis is F_o = {12,134,245,1356}. The diagram of S is as follows:



and hence the game (X,F) is not complete. Its group reduces to be Aut(X,F)={id}, and then orbits and classes coincide, all of them being singletons. However, p(3) = .1 = p(5), in spite of 3 and 5 not being equivalent. Thus, the transposition t_{35} preserves power index but is not an automorphism.

Counterexample 2. Let X ={1,2,...,n}, $n \geqslant 5$, q =[n/2] + 1, and F the filter generated by

$$F_o = \{12...q,\ 23...(q+1),...,\ n1...(q-1)\}$$

(circular subsets of q elements).

Here the group is Aut(X,F) = D_{2n}, the dihedral group, since it contains the rotation $1 \to 2 \to 3 \to \ldots \to n \to 1$, and the symmetry given by s(i) = n-i+1 for each $i \in X$. There are no transpositions, and so there are n symmetry classes (singletons) but only one orbit, the group being transitive. Power index is constant.

Obviously, S is not total: only xSx for each $x \in X$ is satisfied.

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