STOCHASTICA Vol. VIII, Nº 2, 1984.

FUZZY RELATION EQUATION UNDER A CLASS OF TRIANGULAR NORMS: A SURVEY AND NEW RESULTS

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ABSTRACT

By substituting the classical lattice operator "min" of the unit real interval with a triangular norm of Schweizer and Sklar, the usual fuzzy relational equations theory of Sanchez can be generalized to wider theory of fuzzy equations. Considering a remarkable class of triangular norms, for such type of equations defined on finite sets, we characterize the upper and lower solutions.

We also characterize the solutions possessing a minimal fuzziness measure of Yager valued with respect to a triangular norm and conorm.

Moreover we discuss the problem of characterization of the approximate solutions of fuzzy equations.

Finally, the role of the equations considered here in creation of a formal framework for copying with fuzziness is illustrated by various examples in some well known schemes in applications of fuzzy set theory.

1980 Mathematics Subject Classification: Primary 94005, Secondary 03E72.

Key words and phrases: triangular norm, fuzzy relation equation, fuzziness measure, fuzzy system, fuzzy number, possibility measure.

1. Introduction.

The fuzzy relation equations, started in 1976 [46], are appeared for investigating theoretical and applicational aspects of fuzzy set theory. The fuzzy relation equations come originally from Boolean relation equations.

An imposition of fuzziness in the structure of equations has introduced a lot of interesting problems that should be solved in order to provide the user with a deep insight into the role of the fuzzy equations in copying with imprecision.

A number of papers on fuzzy relation equations is significant. Moreover a lot of inter- is paid to various aspects that may be directly used in pract

The aim of this p is to show the state of the art of fuzzy relation equations presenting the main streams of their theoretical and applicational development.

Let be $X=\{x_1,x_2...x_n\}$, $Y=\{y_1,y_2...y_m\}$, $Z=\{z_1,z_2...z_r\}$ finite sets, $F(X \times Y)=\{Q: X \times Y \to [0,1]\}$, $F(Y \times Z)=\{R: Y \times Z \to [0,1]\}$ and $F(X \times Z)=\{T: X \times Z \to [0,1]\}$ the sets of all fuzzy relations defined on the respective spaces. In this famous paper, Sanchez [46] considered th- - -ving fuzzy equation:

$$T = R \circ Q, \quad T(x_i,z_k)=(R \circ Q)(x_i,z_k) = \bigvee_{j=1}^{m} [Q(x_i,y_j) \wedge R(y_j,z_k)] \qquad (1)$$

for any $i \in I_n$, $j \in I_m$, $k \in I_r$ denoting I_n the set of first n natural numbers and $a \wedge b$ (resp. $a \vee b$) the smaller (resp. the greater) between a and b, $a, b \in [0,1]$.

Now let $F(X)=\{A: X \to [0,1]\}$, $F(Y)=\{B: Y \to [0,1]\}$ the sets of all fuzzy sets of X and Y. In a wellknown work, Sanchez [47] introduced also a semplified version of the equation (1), precisely the following:

$$B = M \circ A, \qquad B(y_j) = \bigvee_{i=1}^{n} [A(x_i) \wedge M(x_i,y_j)] \qquad (2)$$

where $M \in F(X \times Y)$, $i \in I_n$, $j \in I_m$..

Pedrycz [39] has shown if we substitute in (1) and (2) the operator "Λ" with a triangular norm, we obtain two more extensive families of fuzzy equations:

$$T=R \,\Box_t\, Q,\quad T(x_i,z_k)=(R \,\Box_t\, Q)(x_i,z_k)=\bigvee_{j=1}^{m} [Q(x_i,y_j)\, t\, R(y_j,z_k)] \tag{3}$$

$$B=M \,\Box_t\, A, B(y_j)=(M \,\Box_t\, A)(y_j)=\bigvee_{i=1}^{n} [A(x_i)\, t\, M(x_i,y_j)] \tag{4}$$

where $i \in I_n$, $j \in I_m$, $k \in I_r$ and $t: [0,1]^2 \rightarrow [0,1]$ is a real function of Schweizer and Sklar [49] having suitable properties specified in section 2.

Through the paper, given $Q \in F(X \times Y)$, $T \in F(X \times Z)$ (resp. $A \in F(X)$, $B \in F(Y)$) we denote with $\mathcal{R}$ (resp. $\mathcal{M}$) the set of the solutions R in $F(Y \times Z)$ (resp. $M \in F(X \times Y)$) of the equation (3) (resp. (4)).

We recall some results already established in foregoing papers.

For the equations (1) and (2), Sanchez [46] determines the greatest element of $\mathcal{R}$ and in [47] characterizes the greatest element of $\mathcal{M}$ and the minimal ones.

In [53], [55], [9], the authors have found the minimal elements of $\mathcal{R}$ under the hypothesis $t \equiv \Lambda$.

By studying the equation (4), Pedrycz [39] defines the greatest element of $\mathcal{M}$ whereas the authors of [12] characterize all the lower solutions, i.e. the minimal elements of $\mathcal{M}$, considering a special class of strictly increasing triangular norms.

In [41] Pedrycz studied a particular equation (4) considering a class of generalized connectives defined by Yager [51].

Miyakoshi and Shimbo [37] have pointed out a detailed study of the equations (3) and (4). Later we shall dedicate wide reference to these authors.

We remember the authors of [35], which give necessary and sufficient conditions in order to guarantee the uniqueness of the solution for the equation (1), i.e. when R has cardinality 1. See also [50] for a similar result about the equation (2).

Here we also signal the contributes of Czogala, Drewniak and Pedrycz [5], Higashi and Klir [29], Pappis and Sugeno [38] for the lower solutions of the equation (2).

We refer to [10], [13], [15] for the equation (1), to [11], [14], [30] for the equation (2), to [12] for the equation (4). In these cited papers the authors minimize the fuzzy entropy of the solutions of a fuzzy equation using the wellknown concepts of De Luca and Termini [7]. Similar results are established by Di Nola and Ventre [16] studying the equation (1) in brouwerian lattices.

Further theoretical results can be quoted in [4], [6], [10], [18],[25],[26],[28], [36], [40], [42], [43], [45], [48], [50], [59], [60].

Existing many theoretical results about the theory of Sanchez, it seems to us quite natural to collect in this survey which deals essentially the equation (3) and (4).

All the results established are slight modifications of the results of Sanchez and our foregoing papers, however new results are given too. Beside every theorem we recall the exact reference where the same result appears, when the theorem is new no reference is cited. Some theorems are also presented with different technical proofs from the usual ones. In every case, the proof is exhibited for sake of completeness.

There are several sections. In section 2 we give the basic preliminaries. In sections 3, 4, 5 we characterize the maximal and minimal elements of M and R.

In sections 6, 7 we characterize the elements of R having the smallest measure of fuzzy entropy of Yager [52] valued by means triangular norms and conorms.

Section 8 deals with approximate solutions of the equations, while section 9 is concerned with some field of applications.

For semplicity, we adopt the notations

$$Q(x_i, y_j) = Q_{ij}, R(y_j, z_k) = R_{jk}, T(x_i, z_k) = T_{ik}, A(x_i)=A_i, B(y_j)=B_j$$

for $i \in I_n$, $j \in I_m$, $k \in I_r$ representing fuzzy sets and fuzzy relations as real matrices.

2. t-norms, conorms and definitions.

Following Schweizer and Sklar [49], a triangular norm (briefly t-norm) is a real function $t:[0,1]^2 \to [0,1]$ of two variables satisfying the following properties:

(2.1)	$0 \; t \; a = a \; t \; 0 = 0, \; 1 \; t \; a = a \; t \; 1 = a$	(boundary conditions)
(2.2)	$a \; t \; b \leqslant a' \; t \; b'$ if $a \leqslant a'$, $b \leqslant b'$	(monotonicity)
(2.3)	$a \; t \; b = b \; t \; a$	(commutativity)
(2.4)	$(atb) \; t \; c = a \; t \; (b \; t \; c)$	(associativity)

where $a, a', b, b', c \in [0,1]$.

Several examples of t-norm can be found in Klement [34], Pedrycz [39, 40]. We cite for any $a, b \in [0,1]$:

$a \; t_1 \; b = a \wedge b$

$a \; t_2 \; b = a \cdot b$

$a \; t_3^{(p)} \; b = 1 - \min\{1, [(1-a)^p + (1-b)^p]^{1/p}\}, \; p \geqslant 1$

$a \; t_4 \; b = \max(0, a + b - 1)$

$a \; t_5^{(\lambda)} \; b = \log_\lambda\{1 + [(\lambda^a - 1) \cdot (\lambda^b - 1)]/(\lambda - 1)\}, \; 0 < \lambda < +\infty, \lambda \neq 1$

$a \; t_6^{(\gamma)} \; b = a \cdot b / [\gamma + (1-\gamma) \cdot (a + b - ab)], \; \gamma \geqslant 0.$

We refer t_1, t_2 to Zadeh [57], $t_3^{(p)}$ to Yager [51], t_4 to Giles [24], $t_5^{(\lambda)}$ to Frank [23], $t_6^{(\gamma)}$ to Hamacher [27] and Alsina, Trillas, and Valverde [2], Aczel's monography [1] for further details. See also Dubois and Prade [19], Weber [54].

Further, the following statements hold [40]:

"$a\ t\ b \leqslant a\ t_1\ b$ for any t-norm and for any $a,b \in [0,1]$",

"for $p=1$, $t_3^{(p)} = t_4$ and $t_3^{(p)} \to t_1$ for $p \to +\infty$" [45],

"$t_5^{(\lambda)} \to t_1$ for $\lambda \to 0$, $t_5^{(\lambda)} \to t_2$ for $\lambda \to 1$, $t_5^{(\lambda)} \to t_4$ for $\lambda \to +\infty$",

"$t_6^{(\gamma)} = t_2$ for $\gamma=1$".

In according to [37], from now on we consider t-norms satisfying the following additional requirement:

(2.5) for any fixed $a \in [0,1]$, $t(a,\cdot)$ is continuous in $[0,1]$.

One can easily prove that the abovecited t-norm satisfy (2.5).

Now we put

$$I_t\ (a,b) = \{x \in [0,1] : a\ t\ x \leqslant b\}$$

where $a,b \in [0,1]$. Like in [37], we define an operator $\psi_t : [0,1]^2 \to [0,1]$ corresponding to the t-norm, as

$$a\ \psi_t\ b = \sup\{x \in I_t(a,b)\}$$

for any $a,b \in [0,1]$.

The assumption (2.5) is necessary in order to assure the belongness of $a\ \psi_t\ b$ to the set $I_t\ (a,b)$, as it shown in the following example indebted to Miyakoshi and Shimbo [37]:

Example 1. Let t_w be the t-norm defined as

$$a\, t_w\, b = \begin{cases} b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

It is immediately seen that if $0<a<1$, $t_w(a,x)$ is not continuous at $x = 1$. Now, if $1>a>b$, we deduce $a\, \psi_w\, b = 1$ being

$$I_{t_w}(a,b) = [0,1[$$

and 1 does not belong to such set because $a\, t_w\, 1 = a>b$.

Obviously $I_t(a,b)$ coincides with the interval $[0,a\psi_t\, b]$ and the following properties hold [37], [40]:

(2.6) $a\, \psi_t\, b \geqslant a\, \psi_t\, c$ if $b \geqslant c$

(2.7) $a t(a\, \psi_t\, b) \leqslant b$

(2.8) $a\, \psi_t\, (a\, t\, b) \geqslant b$

where $a,b,c \in [0,1]$.

From now on, since misunderstanding can arise, we write $a\psi b$ instead of $a\, \psi_t\, b$ for any $a,b \in [0,1]$.

We explicitly note that

$a \leqslant b$ if and only if $a\psi b = 1$

$a\, \psi\, a = 1$

$a\, \psi\, 0 = 0$

$1\, \psi\, a = a$

where $a,b \in [0,1]$.

We list the operators deduced from the abovecited t-norms:

$$a\ \psi_1\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{if } a > b \end{cases}$$

$$a\ \psi_2\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ b/a & \text{if } a > b \end{cases}$$

$$a\ \psi_3^{(p)}\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ 1-[(1-b)^p-(1-a)^p]^{1/p} & \text{if } a>b, p \geqslant 1 \end{cases}$$

$$a\ \psi_4\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ 1+b-a & \text{if } a > b \end{cases}$$

$$a\ \psi_5^{(\lambda)}\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ \log_\lambda\{1+[(\lambda^b-1)\cdot(\lambda-1)]/(\lambda^a-1)\} & \text{if } a>b,\ o<\lambda<+\infty,\ \lambda\neq 1 \end{cases}$$

$$a\ \psi_6^{(\gamma)}\ b = \begin{cases} 1 & \text{if } a \leqslant b \\ [b\gamma+ab\cdot(1-\gamma)]/[a-b+b\gamma+ab(1-\gamma)] & \text{if } a>b,\ \gamma \geqslant 0 \end{cases}$$

where $a,b \in [0,1]$. See Pedrycz [39],[40] also.

We remember some definitions and results of [37].

Let us define for any $a,b \in [0,1]$:

$$G_t(a,b) = \{x \in [0,1]:\ a\ t\ x = b\}$$

Lemma 2.1. [37]. $G_t(a,b) \neq \emptyset$ if and only if $a \geqslant b$.

Proof. Trivial.

Lemma 2.2. [37]. If $G_t(a,b) \neq \emptyset$, then $a\psi b \in G_t(a,b)$.

Proof. Clearly

$$G_t(a,b) \subseteq I_t(a,b)$$

with $a,b \in [0,1]$.

Let $x_o \in G_t(a,b)$. Hence $x_o \leqslant a\psi b$ because x_o is in $I_t(a,b)$ too. Since t is non-decreasing, we have from (2.6) and (2.7):

$$b = atx_o \leqslant a\ t(a\psi b) \leqslant b$$

and thus $a\ t(a\psi b) = b$, i.e. $a\psi b \in G_t(a,b)$.
Obviously

$$a\psi b = \sup\{x \in G_t(a,b)\}$$

and if we put

$$a\beta b = \inf\{x \in G_t(a,b)\}$$

we have, by property (2.5), that

$$a\beta b \in G_t(a,b)$$

and therefore

$$G_t(a,b) = [a\beta b,\ a\psi b].$$

If a t-norm is strictly increasing on the support set

$$\{x \in [0,1] : atx>0\}$$

for any $a>0$, then

$$G_t(a,b) = \{a\beta b\} = \{a\psi b\} \qquad (6)$$

for $a>b>0$.

Example 2. [37]. Let $t = t_1$, thus ψ_1 is the wellknown pseudo-complement of a in b [46]. Then

$$\text{for } a = b,\ I_t(a,a) = [0,1] \text{ and } G_t(a,a) = [a,1],$$

$$\text{for } a>b>0,\ I_t(a,b) = [0,b] \text{ and } G_t(a,b) = \{b\}.$$

In general, we can say that the t-norm t_i, $i \in I_6$, above listed, being strictly increasing, satisfy (6) for $a>b>0$.

If t is a t-norm, the function $s: [0,1]^2 \to [0,1]$ defined by $asb=1 - [(1-a)\ t\ (1-b)]$ for every $a,b \in [0,1]$ is called t-conorm or dual of T. It is immediately proved that any t-conorm satisfies monotonicity, commutativity, associativity properties and the following boundary conditions:

$$0\ s\ a = a\ s\ 0 = a, \quad a\ s\ 1 = 1\ s\ a = 1 \quad \text{for any } a \in [0,1].$$

The t-conorms obtained from abovecited t-norms are respectively:

$a\ s_1\ b = a \vee b$

$a\ s_2\ b = a+b-ab$

$a\ s_3^{(p)} b = \min[1,(a^p+b^p)^{1/p}],\ p \geqslant 1$

$a\ s_4\ b = \min(1,a+b)$

$a\ s_5^{(\lambda)} b = 1-\log_\lambda\{1+[(\lambda^{(1-a)}-1)\cdot(\lambda^{(1-b)}-1)]/(\lambda-1)\},\ 0<\lambda<+\infty,\ \lambda\neq 1$

$a\ s_6^{(\gamma)} b = [a b.(\gamma-2)+a+b]/[a.b(\gamma-1)+1],\ \gamma \geqslant 0$

with $a,b \in [0,1]$.

From (5), we deduce:

"$a \vee b \leqslant a\ s\ b$ for any t-conorm and for any $a,b \in [0,1]$".

and let us define inductively for $a_1, a_2, \ldots a_n \in [0,1]$:

$s[a_1,a_2] = a_1 \; s \; a_2,$

$s[a_1,a_2,a_3] = a_1 \; s(s[a_1,a_2]),$

......

$s[a_1,a_2,\ldots a_n] = a_1 \; s(s[a_2,a_3,\ldots a_n]) = \underset{i=1}{\overset{n}{s}} [a_i].$

Now we remember the following definitions [46]:

Definition 2.1. A fuzzy relation $M \in F(X \times Y)$ (resp. fuzzy set $A \in F(X)$) is contained in a fuzzy relation $M' \in E(X \times Y)$ (resp. $A' \in F(X)$) if $M_{ij} \leqslant M'_{ij}$ (resp. $A_i \leqslant A'_i$) for any $i \in I_n$, $j \in I_m$ (resp. $i \in I_n$).

Definition 2.2. We define inverse of $Q \in F(X \times Y)$, the fuzzy relation $Q^{-1} \in F(Y \times X)$ given by $Q^{-1}_{ji} = Q_{ij}$ for any $i \in I_n$, $j \in I_m$.

In according to Pedrycz [40], we give also:

Definition 2.3. Let t be a t-norm, $Q \in F(X \times Y)$, $R \in F(Y \times Z)$. We define $T = R \,\square_t\, Q$, $T \in F(X \times Z)$, given by (3) the sup-t composition of R and Q.

If $t = \Lambda$, of course we have the wellknown max-min composition.

Definition 2.4 The sup-t composition between a fuzzy set $A \in F(X)$ and a fuzzy relation $M \in F(X \times Y)$ is the fuzzy $B \in F(Y)$ given by (4).

Definition 2.5. The $\textcircled{\psi}$-composition of the fuzzy relation $Q^{-1} \in F(YxX)$ and $T \in F(X \times Z)$ is the fuzzy relation $Q^{-1} \textcircled{\psi} T \in F(Y \times Z)$ given by

$$(Q^{-1} \textcircled{\psi} T)\;(y_j, z_k) = \bigwedge_{i=1}^{n} (Q^{-1}_{ji} \;\psi\; T_{ik})$$

for every $j \in I_m, k \in I_r$.

It $t = \Lambda$, then def. 2.5 becomes def. 11 of Sanchez [46].

Definition 2.6. The $\textcircled{\psi}$-composition of two fuzzy sets $A \in F(X)$, $B \in F(Y)$ is the fuzzy relation $A \textcircled{\psi} B \in F(X \times Y)$ given by $(A \textcircled{\psi} B)(x_i, y_j) = A_i \; \psi \; B_j$ for any $i \in I_n$, $j \in I_m$.

Definition 2.7. We define upper (lower) solution of the equation (3) or (4) a maximal (minimal) element of the set $\mathcal{R}$ or $\mathcal{M}$.

3. Resolution of sup-t fuzzy equations.

The results of this Section and Section 4 are presented following essentially the contents of our previous papers.

The same results are established in [37] with slightly different technical proofs. For convenience of the reader, we give all the proofs.

By modifying slightly the results of [46], we give a fundamental theorem for the resolution of equations (3) and (4). We first show some theorems:

Theorem 3.1. [12]. If $R_1, R_2 \in F(Y \times Z)$ and $R_1 \leqslant R_2$, then $R_1 \square_t Q \leqslant R_2 \square_t Q$.

Proof. Trivial by def. 2.3 and monotonicity property of the norm t.

Theorem 3.2. [40]. For any $R \in F(Y \times Z)$, $Q \in F(X \times Y)$ it is

$$R \leqslant Q^{-1} \textcircled{\psi} (R \square_t Q)$$

Proof. By putting $U = Q^{-1} \textcircled{\psi} (R \square_t Q)$, we have for any $j \in I_m$, $k \in I_r$:

$$U_{jk} = \bigwedge_{i=1}^{n} [Q_{ij}\psi(R \,\square_t\, Q)_{ik}] = \bigwedge_{i=1}^{n} [Q_{ij}\psi(\bigwedge_{i=1}^{n} Q_{i1} \; t \; R_{1k})] =$$

$$\bigwedge_{i=1}^{n} \{Q_{ij}\psi[(Q_{ij} \; t \; R_{jk}) \vee (\bigvee_{1\neq j} (Q_{i1} \; t \; R_{1k}))]\}.$$

By properties (2.6) and (2.8), we deduce for any $j \in I_m$, $k \in I_r$:

$$U_{jk} \geqslant \bigwedge_{i=1}^{n} [Q_{ij} \; \psi(Q_{ij} \; t \; R_{jk})] \geqslant R_{jk}$$

and therefore $U \geqslant R$, i.e. the thesis.

<u>Theorem 3.3.</u>[40]. For every $Q \in F(X \times Y)$, $T \in F(X \times Z)$ it is:

$$(Q^{-1} \circledS T) \,\square_t\, Q \leqslant T$$

Proof. By setting $S = (Q^{-1} \circledS T) \,\square_t\, Q$, from monotonicity of the t-norm, we have for every $i \in I_n$, $k \in I_r$:

$$S_{ik} = \bigvee_{j=1}^{m} [Q_{ij} \; t(Q^{-1} \circledS T)_{jk}] = \bigvee_{j=1}^{m} [Q_{ij} \; t(\bigwedge_{i=1}^{n} (Q_{ij} \; \psi \; T_{ik}))] =$$

$$\bigvee_{j=1}^{m} \{Q_{ij} \; t[(Q_{ij} \; \psi \; T_{ik}) \wedge (\bigwedge_{h\neq i} (Q_{hj} \; \psi \; T_{hk}))]\} \leqslant$$

$$\bigvee_{j=1}^{m} [Q_{ij} \; t(Q_{ij} \; \psi \; T_{ik})].$$

Then (2.7) implies the thesis being $S_{ik} \leqslant T_{ik}$ for every $i \in I_n$, $k \in I_r$.

Now we are able to show the following fundamental theorem for the equation (3):

<u>Theorem 3.4 [40].</u> $\mathcal{R} \neq \emptyset$ if and only if $Q^{-1} \circledS T \in \mathcal{R}$ and $Q^{-1} \circledS T \geqslant R$ for every $R \in \mathcal{R}$.

Proof. If $\mathcal{R} \neq \emptyset$, we have by Theorem 3.2.:

$$R \leqslant Q^{-1} ⓟ (R \,\square_t\, Q) = Q^{-1} ⓟ T$$

for any $R \in \mathcal{R}$. From Theorems 3.1 and 3.3, it follows:

$$T = R \,\square_t\, Q \leqslant (Q^{-1} ⓟ T) \,\square_t\, Q \leqslant T$$

and this implies that $Q^{-1} ⓟ T \in \mathcal{R}$. The converse implication is obvious.

Analogously, by using def. 2.4 and 2.6, one can prove the following fundamental theorem:

<u>Theorem 3.5</u> [40]. $\mathcal{M} \neq \emptyset$ if and only if $A ⓟ B \in \mathcal{M}$ and $A ⓟ B \geqslant M$ for every $M \in \mathcal{M}$.

<u>Theorem 3.6.</u> [46]. Let $\mathcal{R} \neq \emptyset$. Then for any $i \in I_n$, $k \in I_r$:

$$T_{ik} \leqslant \bigvee_{j=1}^{m} Q_{ij} \text{ and } T_{ik} \leqslant \bigvee_{j=1}^{m} R_{jk}$$

Proof. The thesis follows from (5) being

$$T_{ik} = \bigvee_{j=1}^{m} (Q_{ij} \; t \; R_{jk}) \leqslant \bigvee_{j=1}^{m} (Q_{ij} \wedge R_{jk})$$

for any $i \in I_n$, $k \in I_r$ whatever is the norm t.

<u>Theorem 3.7.</u> If $R_1, R_2 \in \mathcal{R}$, $R \in F(Y \times Z)$ and $R_1 \leqslant R \leqslant R_2$, then $R \in \mathcal{R}$.

Proof. Trivial.

By defining for any $i \in I_n$, $j \in I_m$, $k \in I_r$:

$$Q_i(x_i, y_j) = Q_{ij} \text{ and } T_i(x_i, z_k) = T_{ik},$$

we deduce that $Q_i \in F(\{x_i\} \times Y)$ and $T_i \in F(\{x_i\} \times Z)$ for any $i \in I_n$. If we interpret Q_i and T_i as fuzzy sets on the respective spaces, we can affirm that the fuzzy equation (3) is equivalent to a system of n fuzzy equations of type (4) rewritten as

$$T_h = Q_h \,\square_t\, R \tag{7}$$

where $h \in I_n$. By denoting with M_i the solutions set of the equation (7) of course we have:

$$R = M_1 \cap M_2 \cap \dots \cap M_n \tag{8}$$

This equality has been pointed out by the authors of [9], [10], [15]. In according to def. 2.6 and theorem 3.5, if we define $Q_i \,\textcircled{\psi}\, T_i$ as the greatest element of M_i, $i \in I_n$, then we have the following:

Theorem 3.8 [26]. If $R \neq \emptyset$, it is

$$Q^{-1} \textcircled{\psi}\, T = \bigwedge_{i=1}^{n} (Q_i \,\textcircled{\psi}\, T_i).$$

Proof. Being $R \neq \emptyset$, it is $M_i \neq \emptyset$ for any $i \in I_n$ too. By theorem 3.4, $Q^{-1} \textcircled{\psi}\, T \in R$ and therefore $Q^{-1} \textcircled{\psi}\, T \in M_i$ for any $i \in I_n$ by (8). So

$$Q^{-1} \textcircled{\psi}\, T \leqslant \bigwedge_{i=1}^{n} (Q_i \,\textcircled{\psi}\, T_i) \leqslant Q_i \,\textcircled{\psi}\, T_i \tag{9}$$

in account of Theorem 3.5. Theorem 3.7 and (9) imply that the fuzzy relation $\bigwedge_{i=1}^{n} (Q_i \,\textcircled{\psi}\, T_i)$ belongs to M_i for any $i \in I_n$ and thus $\bigwedge_{i=1}^{n} (Q_i \,\textcircled{\psi}\, T_i)$ lies in R since (8) holds.

By the fundamental Theorem 3.4, we obtain:

$$\bigwedge_{i=1}^{n} (Q_i \,\textcircled{\psi}\, T_i) \leqslant Q^{-1} \textcircled{\psi}\, T$$

which guarantees the thesis.

Example 3. Let $n=m=r=3$, $Q \in F(X \times Y)$, $T \in F(X \times Z)$ defined as

$$Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.8 & 0.0 & 0.4 \\ 0.5 & 0.1 & 0.3 \\ 1.0 & 0.2 & 0.1 \end{pmatrix} \end{array} \qquad T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.5 & 0.7 & 0.2 \\ 0.2 & 0.4 & 0.1 \\ 0.7 & 0.9 & 0.2 \end{pmatrix} \end{array}$$

By choosing t_4 as t-norm, we have

$$Q^{-1} \textcircled{ψ_4} T = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 1.0 & 1.0 & 1.0 \\ 0.9 & 1.0 & 0.8 \end{pmatrix} \end{array}$$

One can ascertain $(Q^{-1} \textcircled{$\psi_4$} T) \square_{t_4} Q = T$ and since

$$Q_1 \textcircled{ψ_4} T_1 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.4 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 0.8 \end{pmatrix} \end{array} \quad Q_2 \textcircled{ψ_4} T_2 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.6 \\ 1.0 & 1.0 & 1.0 \\ 0.9 & 1.0 & 0.8 \end{pmatrix} \end{array} \quad Q_3 \textcircled{ψ_4} T_3 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{pmatrix} \end{array}$$

it is easily verified Theorem 3.8.

Concluding this section, we can say the above results assure the existence and the uniqueness of an upper solution of the equations (3) and (4).

4. Lower solutions of equation (4).

In this section we characterize the lower solutions of the equation (4) whose existence implies the existence of the lower solutions of the equation (3), as we shall prove in section 5. Here we report some theorems established in [12].

Following [47], we define for every $j \in I_m$:

$$\Gamma_j = \{ i \in I_n \text{ such that } A_i \geqslant B_j \}.$$

Then holds the following

<u>Theorem 4.1.</u>[12]. If $\mathcal{M} \neq \emptyset$, then $\Gamma_j \neq \emptyset$ for any $j \in I_m$.

Proof. From (5), we have for any $j \in I_m$ and $M \in \mathcal{M}$:

$$B_j = \bigvee_{i=1}^{n} (A_i \ t \ M_{ij}) \leqslant \bigvee_{i=1}^{n} (A_i \wedge M_{ij}) \leqslant \bigvee_{i=1}^{n} A_i \tag{10}$$

By putting for any $j \in I_m$:

$$A_{i(j)} = \bigvee_{i=1}^{n} A_i,$$

from (10) it follows that $i(j) \in \Gamma_j$, i.e. $\Gamma_j \neq \emptyset$.

<u>Theorem 4.2.</u> [12]. If $\mathcal{M} \neq \emptyset$, then $\mathcal{M}$ has minimal elements L whose membership functions are defined pointwise choosing, for any $j \in I_m$, an index $i(j) \in \Gamma_j$ such that

$$B_j = A_{i(j)} \ t(A_{i(j)} \beta B_j)$$

and putting for any $i \in I_n$, $j \in I_m$:

$$L_{ij} = \begin{cases} A_{i(j)} \beta B_j & \text{if } B_j > 0, \ i = i(j) \\ 0 & \text{if } B_j > 0, \ i \neq i(j) \\ 0 & \text{if } B_j = 0 \end{cases}$$

Proof. Let $i(j) \in \Gamma_j$, which is non-empty for any $j \in I_m$ (see Theorem 4.1). So $A_{i(j)} \geqslant B_j$ and therefore $G_t(A_{i(j)}, B_j) \neq \emptyset$ by Lemma 2.1. Since

$$A_{i(j)} \ t \ (A_{i(j)} \beta B_j) = B_j,$$

it is easily seen that $L \in \mathcal{M}$ and let $M \in \mathcal{M}$ such that $M \leqslant L$. We have

$$M_{ij} = L_{ij} = 0$$

for any $j \in I_m$ such that either $B_j = 0$ or $B_j > 0$ and $i \neq i(j)$, whereas we have for any $j \in I_m$ such that $B_j > 0$:

$$B_j = \bigvee_{i=1}^{n} A_i \ t \ M_{ij} = [\bigvee_{i \neq i(j)} (A_i \ t \ M_{ij})] \vee (A_{i(j)} \ t \ M_{i(j)j}) =$$

$$0 \vee (A_{i(j)} t \ M_{j(j)j}) = A_{i(j)} \ t \ M_{i(j)j},$$

i.e. $M_{i(j)j} \in G_t(A_{i(j)}, B_j)$ and so

$$L_{i(j)j} = A_{i(j)} \beta \ B_j \leqslant M_{i(j)j} \leqslant L_{i(j)j}.$$

This implies $L = M$, i.e. L is minimal in $\mathcal{M}$. Now we show that an arbitrary lower solution M of $\mathcal{M}$ is a fuzzy relation of the type, as defined in the statement.

Now we have for any $j \in I_m$ such that $B_j > 0$:

$$0 < B_j = \bigvee_{i=1}^{n} (A_i \ t \ M_{ij}) =$$

$$[\bigvee_{i \in \Gamma_j} (A_i \ t \ M_{ij})] \vee [\bigvee_{i \in \Gamma_j} (A_i \ t \ M_{ij})] \qquad (11)$$

being $\Gamma_j \neq \emptyset$ for any $j \in I_m$ by Theorem 4.1.

Since

$$A_i \ t \ M_{ij} \leqslant A_i \ t \ 1 = A_i < B_j$$

for any $i \in \Gamma_j$, (11) becomes

$$0 < B_j = \bigvee_{i \in \Gamma_j} (A_i \ t \ M_{ij}).$$

Then there exists an index $h \in \Gamma_j$ such that

$$B_j = A_n \ t \ M_{hj},$$

i.e. $M_{hj} \geqslant A_h \beta B_j$. Therefore, assuming $h=i(j)$ for any $j \in I_m$ such that $B_j>0$, we can build a fuzzy relation $L \in \mathcal{M}$ as defined in the statement, such that

$$L_{i(j)j} \leqslant M_{i(j)j}$$

if $B_j>0$ and $0=L_{ij} \leqslant M_{ij}$ if either $B_j=0$ or $B_j>0$ and $i \neq i(j)$. So such relation L is contained in M and so L = M being M minimal in $\mathcal{M}$. This concludes the proof.

Now let us define the fuzzy relation $A \circledbeta B \in F(X \times Y)$ as

$$(A \circledbeta B)_{ij} = \begin{cases} A_i \ \beta\ B_j & \text{if } A_i \geqslant B_j > 0 \\ 0 & \text{if } A_i < B_j \neq 0 \\ 0 & \text{if } B_j = 0 \end{cases}$$

for any $i \in I_n$, $j \in I_m$.

Let us denote [14] with $J = \{j \in I_m \text{ such that } B_j = 0\}$. Then it is easy to show the following:

Theorem 4.3. If $\Gamma_j \neq \emptyset$ for any $j \in I_m$, then $A \circledbeta B \in \mathcal{M}$.

Proof. For any $j \in I_m - J$, we have from property (2.1):

$$\bigvee_{i=1}^{n} [A_i \ t (A \circledbeta B)_{ij}] = \{ \bigvee_{i \in \Gamma_j} [A_i \ t (A_i \ \beta\ B_j)]\} \vee \{ \bigvee_{i \notin \Gamma_j} (A_i \ t \ 0)\} =$$

$$\bigvee_{i \in \Gamma_j} \{A_i \ t \ (A_i \ \beta\ B_j)\} = B_j .$$

If $j \in J$, the thesis is straightforward.

Theorem 4.4. If $\mathcal{M} \neq \emptyset$, the fuzzy union of all minimal elements of $\mathcal{M}$ is $A \circledbeta B$.

Proof. It is evident from definitions of $A \circledbeta B$ and lower solution.

Theorem 4.5. [14]. The number μ of all lower solutions of $\mathcal{M}$ is given by

$$\mu = \prod_{j \in I_m - J} \text{card } \Gamma_j$$

Proof. The thesis follows from definition of lower solution. In other words, in order to determine a lower solution L, it suffices to keep a non-zero element $A_i \ \beta \ B_j$ in the j-th column of $A \circledbeta B$ for which $B_j > 0$, $j \in \Gamma_j$ and this can be made only in μ possible ways. If $B_j = 0$, one obviously assumes $L_{ij} = 0$ for any $i \in I_n$.

<u>Example 4.</u> [12]. Let be m=n=3, $A \in F(X)$, $B \in F(Y)$ given by

$$A = \begin{matrix} x_1 & x_2 & x_3 \\ [0.5 & 1.0 & 0.9] \end{matrix} \qquad B = \begin{matrix} y_1 & y_2 & y_3 \\ [0.5 & 0.0 & 0.6] \end{matrix}$$

considering the norm $t_3^{(2)}$, we have $\Gamma_1 = \Gamma_2 = I_3$, $\Gamma_3 = \{2,3\}$ and

$$(A\ \psi_3^{(2)} B) = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 1.00 & 0.14 & 1.00 \\ x_2 & 0.50 & 0.00 & 0.60 \\ x_3 & 0.52 & 0.01 & 0.62 \end{matrix} \qquad (A\psi_3^{(2)} B) = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 1.00 & 0.00 & 0.00 \\ x_2 & 0.50 & 0.00 & 0.60 \\ x_3 & 0.52 & 0.00 & 0.62 \end{matrix}$$

Being = 6, we have 6 lower solutions given by

$$\begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 1.00 & 0.00 & 0.00 \\ x_2 & 0.00 & 0.00 & 0.60 \\ x_3 & 0.00 & 0.00 & 0.00 \end{matrix} \qquad \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.00 & 0.00 & 0.00 \\ x_2 & 0.50 & 0.00 & 0.60 \\ x_3 & 0.00 & 0.00 & 0.00 \end{matrix}$$

$$\begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.00 & 0.00 & 0.00 \\ x_2 & 0.00 & 0.00 & 0.60 \\ x_3 & 0.52 & 0.00 & 0.00 \end{matrix} \qquad \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 1.00 & 0.00 & 0.00 \\ x_2 & 0.00 & 0.00 & 0.00 \\ x_3 & 0.00 & 0.00 & 0.62 \end{matrix}$$

$$
\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{ccc} y_1 & y_2 & y_3 \\ \left(\begin{array}{ccc} 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.62 \end{array}\right) \end{array}
\qquad
\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{ccc} y_1 & y_2 & y_3 \\ \left(\begin{array}{ccc} 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.52 & 0.00 & 0.62 \end{array}\right) \end{array}
$$

<u>Theorem 4.6.</u> $A \text{\textcircled{$\beta$}} B$ is the minimum of M if only if card $\Gamma_j = 1$ or $B_j = 0$ for any $j \in I_m$.

Proof. If $J = I_n$, i.e. $B_j = 0$ for any $j \in I_m$, then $(A \text{\textcircled{$\beta$}} B)_{ij} = L_{ij} = 0$ for any $i \in I_n$, $j \in I_m$ and clearly $A \text{\textcircled{$\beta$}} B$ is the minimum of M. Therefore, let us suppose $J \neq I_m$, i.e. $B_j > 0$ for some $j \in I_m$. Since it is card $\Gamma_j = 1$ for any $j \in I_m - J$, we have $\mu = 1$ from theorem 4.5, i.e. $A \text{\textcircled{$\beta$}} B$ is the unique lower solution of M and this proves the sufficiency.

Viceversa, let $A \text{\textcircled{$\beta$}} B$ be the minimum of M. This implies card $\Gamma_j = 1$ for any $j \in I_m - J$.

The proof, here presented, is quite different from one indebted to Sanchez [47].

<u>Example 5.</u> Let be $n=3$, $m=4$, $A \in F(X)$, $B \in F(Y)$ defined as

$$
A = \begin{array}{ccc} x_1 & x_2 & x_3 \\ [\,0.2 & 0.8 & 0.5\,] \end{array}
\qquad
B = \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ [\,0.7 & 0.6 & 0.0 & 0.8\,] \end{array}
$$

Note that $\Gamma_1 = \Gamma_2 = \Gamma_4 = \{2\}$, $J = \{3\}$. By using the norm t_2, we have:

$$
\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ \left(\begin{array}{cccc} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.875 & 0.750 & 0.000 & 1.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{array}\right) \end{array}
= A \text{\textcircled{ψ_2}} B = A \text{\textcircled{β_2}} B
$$

<u>Theorem 4.7</u> If $M \neq \emptyset$, for any $M \in M$ there exists a lower solution $L \in M$ such that $L \leqslant M$.

Proof. The thesis is obvious if $I_m = J$. Then let us suppose $B_j > 0$ for some $j \in I_m \neq J$. Since we have:

$$0<B_j = \bigvee_{i=1}^{n} (A_i \; t \; M_{ij})$$

for any $M \in \mathcal{M}$, there exists an index $i(j) \in I_n$ such that $B_j = A_{i(j)} \; t M_{i(j)j} \leqslant A_{i(j)}$. This means $i(j) \in \Gamma_j$ and thus $M_{i(j)} \geqslant A_{i(j)} \beta B_j$.

Then by defining for any $i \in I_n$, $j \in I_m$:

$$L_{ij} = \begin{cases} 0 & \text{if } j \in J \\ A_{i(j)} \beta \; B_j & \text{if } i=i(j),\; j \in I_m - J \\ 0 & \text{if } i \neq i(j),\; j \in I_m - J, \end{cases}$$

we obtain a lower solution $L \leqslant M$.

5. Lower solutions of equation (3).

Since the study of max-t fuzzy equations (3) is transferred to the study of max-t fuzzy equations (4), we can apply the results of section 4 for determining the lower solutions of equation (3).

First we present the following:

Theorem 5.1. [9]. For any $h \in I_n$, if $\mathcal{R} \neq \emptyset$, there exists a lower solution $L^{(h)}$ of the equation (7) such that $L^{(h)} \leqslant Q^{-1} \circledpsi T$.

Proof. By interpreting the equation (7) as a fuzzy equation of type (4), recalling the formula (8), we have that $\mathcal{M} \neq \emptyset$ for any $h \in I_n$ and furtherly $Q^{-1} \circledpsi T$ belongs to $\mathcal{M}_h$ for any $h \in I_n$. By invoking Theorem 4.7, there exists a lower solution $L^{(h)}$ of equation (7) such that $L^{(h)} \leqslant Q^{-1} \circledpsi T$ for every $h \in I_n$.

Denoting with $\mathcal{L}^{(h)}$, $h \in I_n$, the set of lower solutions $L^{(h)} \in \mathcal{M}_h$ contained in $Q^{-1} \circledpsi T$, let $\mathcal{L}$ be the set given by

$$\mathcal{L} = \{L = \bigvee_{h=1}^{n} L^{(h)},\; L^{(h)} \in \mathcal{L}^{(h)},\; h \in I_n\},$$

i.e. an element of $\mathcal{L}$ is the fuzzy union of n elements $L^{(h)}$, each chosen in the set $\mathcal{L}^{(h)}$ for any $h \in I_n$.

Since obviously we have

$$L^{(h)} \leqslant L = \bigvee_{h=1}^{n} L^{(h)} \leqslant Q^{-1} \circledS T,$$

consequently, by Theorem 3.7, $\mathcal{L}$ is a finite subset of each $\mathcal{M}_h, h \in I_n$, and thus it is a subset of $\mathcal{R}$ by (8).

By using the wellknown concept that every finite set has minimal elements, we are in position to prove the following characterization of the minimal elements of the equation (3):

Theorem 5.2. [9]. If $\mathcal{R} \neq \emptyset$, the minimal elements of $\mathcal{L}$ are the minimal elements of $\mathcal{R}$ and vice-versa.

Proof. We only prove the non trivial implication: $\bar{L}$ minimal in $\mathcal{L}$, then $\bar{L}$ minimal in $\mathcal{R}$. Indeed, let be $R \in \mathcal{R}$ such that $R \leqslant \bar{L}$. By equality (8), the hypothesis $\mathcal{R} \neq \emptyset$ implies $\mathcal{M}_h \neq \emptyset$ for any $h \in I_n$ and then there exists a lower solution $L^{(h)} \in \mathcal{M}_h$ such that $L^{(h)} \leqslant R$. Being $R \leqslant Q^{-1} \circledS T$ (because $Q^{-1} \circledS T$ is the greatest element of $\mathcal{R}$), we deduce that $L^{(h)} \in \mathcal{L}_h$ for every $h \in I_n$. So the fuzzy relation $L \in \mathcal{L}$ given by the fuzzy union of such elements $L^{(h)} \in \mathcal{L}_h$ verifies the inequality:

$$L = \bigvee_{h=1}^{n} L^{(h)} \leqslant R \leqslant \bar{L} \tag{12}$$

But the minimality of $\bar{L}$ in $\mathcal{L}$ implies $L=\bar{L}$ and therefore $\bar{L}=R$ by (12).

We illustrate the results of this section with the following

Example 6. Recalling example 3, we have

$$Q_1 \textcircled{\beta_4} T_1 = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.4 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.0 & 0.8 \end{matrix} \qquad Q_2 \textcircled{\beta_4} T_2 = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.6 \\ y_2 & 0.0 & 0.0 & 1.0 \\ y_3 & 0.9 & 0.0 & 0.8 \end{matrix}$$

$$Q_3 \textcircled{\beta_4} T_3 = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.2 \\ y_2 & 0.0 & 0.0 & 1.0 \\ y_3 & 0.0 & 0.0 & 0.0 \end{matrix}$$

Bearing in mind theorems 4.2 and 4.4. M_1 has the following lower solutions:

$$L_1^{(1)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.0 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.0 & 0.8 \end{matrix} \qquad L_2^{(1)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.4 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.0 & 0.0 \end{matrix}$$

M_2 contains six lower solutions given by

$$L_1^{(2)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.6 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.0 & 0.0 \end{matrix} \qquad L_2^{(2)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.0 \\ y_2 & 0.0 & 0.0 & 1.0 \\ y_3 & 0.0 & 0.0 & 0.0 \end{matrix}$$

$$L_3^{(2)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.7 & 0.9 & 0.0 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.0 & 0.0 & 0.8 \end{matrix}$$

$$L_4^{(2)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.0 & 0.9 & 0.6 \\ y_2 & 0.0 & 0.0 & 0.0 \\ y_3 & 0.9 & 0.0 & 0.0 \end{matrix} \qquad L_5^{(2)} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.0 & 0.9 & 0.0 \\ y_2 & 0.0 & 0.0 & 1.0 \\ y_3 & 0.9 & 0.0 & 0.0 \end{matrix}$$

$$L_6^{(2)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.0 & 0.9 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

M_3 has two lower solutions given by

$$L_1^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix} \end{array} \qquad L_2^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix} \end{array}$$

Consequentely, $\mathcal{L}^{(1)}=\{L_1^{(1)}\}$, $\mathcal{L}^{(2)}=\{L_2^{(2)}, L_3^{(2)}, L_5^{(2)}, L_6^{(2)}\}$, $\mathcal{L}^{(3)}=\{L_1^{(3)}, L_2^{(3)}\}$.

So $\mathcal{L}$ possesses eight elements defined as

$$L = L_1^{(1)} \vee L_2^{(2)} \vee L^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array} \qquad L_2 = L_1^{(1)} \vee L_2^{(2)} \vee L_2^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.1 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

$$L_3 = L_1^{(1)} \vee L_3^{(2)} \vee L_1^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array} \qquad L_4 = L_1^{(1)} \vee L_3^{(2)} \vee L_2^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

$$L_5 = L_1^{(1)} \vee L_5^{(2)} \vee L_1^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 0.0 & 0.0 & 0.1 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \end{array} \qquad L_6 = L_1^{(1)} \vee L_5^{(2)} \vee L_2^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

$$L_7 = L_1^{(1)} \vee L_6^{(2)} \vee L^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \end{array} \qquad L_8 = L_1^{(1)} \vee L_6^{(2)} \vee L_2^{(3)} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.9 & 0.0 & 0.8 \end{pmatrix} \end{array}$$

where $L_2=L_4, L_6=L_8$ and therefore L has exactly six distinct elements such that $L_4 \leqslant L_1 \leqslant L_5$, $L_4 \leqslant L_6 \leqslant L_5$,

$$L_3 \leqslant L_1, \quad L_3 \leqslant L_5, \quad L_3 \leqslant L_7, \quad L_3 \nleqslant L_4.$$

Then L_3 and L_4 are the lower solutions of L.

6. Fuzziness measures based on t-norms and t-conorms.

By using the concept of t-norm and t-conorm, Yager [52] defines a pseudo-metric on F(X), $X = \{x_1, x_2 \ldots x_n\}$, which induces a class of measures of fuzziness associated with a fuzzy set $A \in F(X)$ as

$$H_{t,s}(A) = \mathop{s}_{j=1}^{n} [A_j \ t(1-A_j)] \tag{13}$$

where s is any t-conorm, which is not necessarily the dual of t. De Luca and Termini [7] point out that any measure of fuzzy entropy H should have the following suitable properties:

(6.1) $H(A)=0$ iff A is a crisp set, i.e. $A_j \in \{0,1\}$ for any $j \in I_n$.

(6.2) If $A_j=1/2$ for any $j \in I_n$, then $H(A) > H(B)$ for any $B \in F(X)$ such that $B \neq A$.

(6.3) If $A, B \in F(X)$, then $H(A) \geqslant H(B)$ if B is sharper than A, in symbols $B \leqslant_{sh} A$, i.e. if for any $j \in I_n$, it is $A_j \leqslant B_j$ when $A_j \geqslant 1/2$ and $A_j \geqslant B_j$ when $A_j \leqslant 1/2$.

Investigating the satisfaction in order to (13) fulfils these axioms, Yager [52] establishes the following theorems and definition:

Theorem 6.1. $H_{t,s}(A)=0$ iff $t > t_4$ or $t \geqslant t_2$ for any t-conorm s.

Theorem 6.2. If $A \in F(X)$ is such that $A_i = 1/2$ for any $i \in I_n$, then for any $B \in F(X)$ for which $B \neq A$, we have:

$$H_{t_2, s_3^{(P)}}(A) > H_{t_2, s_3^{(p)}}(B)$$

if $p \in [\log_2 n/2, +\infty]$ for $n \geq 4$ and if $p \geq 1$ for $n<4$,

$$H_{t_3^{(q)}, s_3^{(p)}}(A) > H_{t_3^{(q)}, s_3^{(p)}}(B)$$

if $t_3^{(q)}$ is an operator from Yager's family of t-norms with parameter $q>1$ and $p \in [1,+\infty] \cap [w,+\infty]$ where $w = -\log_2 n / \log_2 [1-(1/2)^{q-1/q}]$. In particular, if $q \to +\infty$ then $w = \log_2 n$ and $t_3^{(q)} \to t_1$ (cfr. section 2).

Definition 6.1. A t-norm is regular under complements if for every $a, b \in [0,1]$:

$$a \wedge (1-a) \geq b \wedge (1-b) \text{ implies } a\ t(1-a) \geq b\ t(1-b) \qquad (14)$$

Theorem 6.3. The t-norms $t_1, t_2, t_3^{(p)}$ for any $p \geq 1$ satisfy (14).

Theorem 6.4. If t is a t-norm regular under complements and s is any t-conorm, $H_{t,s}$ (given by (13)) satisfies the axiom (6.3).

We are also interested to the t-norms verifying strictly (14), i.e. for any $a, b \in [0,1]$:

$$a \wedge (1-a) > b \wedge (1-b) \text{ implies } a\ t(1-a) > b\ t(1-b). \qquad (15)$$

Definition 6.2. We call strictly regular under complements the t-norms regular under complements verifying (15).

Lemma 6.5. If $b>a>1/2$ or $b<a<1/2$, then $a\ t(1-a) > b\ t(1-b)$ if $t=t_1$ or $t=t_2$ or $t=t_3^{(p)}$ with $p>1$.

Proof. The thesis is obvious for t_1. Let us consider the function so defined for every $a,b\in[0,1]$:

$$f(a,b) = a-a^2-b+b^2$$

Since $\partial f/\partial a=1-2a$, for $b>a>1/2$ we have $\partial f/\partial a<0$. Being $f(b,b)=0$ and $(a,b)<(b,b)$, we deduce $f(a,b)>0$ and this implies the thesis. If $b<a<1/2$, then $\partial f/\partial a>0$ and, being $(b,b)<(a,b)$, we have again $f(a,b)>0$. So the thesis is true for $t=t_2$. In connection with $t=t_3^{(p)}$, we consider the function given by

$$g(a,b)=a^p + (1-a)^p - b^p - (1-b)^p,\ p>1$$

for every $a,b\in[0,1]$. Arguing as in the Lemma at page 220 of [52], also we have $\partial g/\partial a=p.a^{p-1}-p(1-a)^{p-1}$. If $b>a>1/2$, then $a>1-a$ and $(a,b)<(b,b)$. Resulting $\partial g/\partial a>0$, we have $g(a,b)<g(b,b)=0$, i.e. $a\ t_3^{(p)}(1-a)>b\ t_3^{(p)}(1-b)$. If $b<a<1/2$, then $a<1-a$ and so $\partial g/\partial a<0$. Being $(b,b)<(a,b)$ we have $g(a,b)<g(b,b)=0$ too, i.e. the thesis.

<u>Lemma 6.6.</u> For $t=t_1$ or $t=t_2$ or $t=t_3^{(p)}$ with $p>1$, we have $1/2\ t\ 1/2>a\ t\ (1-a)$ for any $0\leqslant a\leqslant 1$, $a\neq 1/2$.

<u>Proof.</u> Trivial for t_1. Since $(1/2-a)^2>0$ for any $a\neq 1/2$, we have $1/2\ t_2\ 1/2 = 1/4>a - a^2 = at_2(1-a)$, i.e. the thesis for t_2. Regarding $t_3^{(p)}$, we have

$$\partial^2 g/\partial a^2 = p\cdot(p-1).[a^{p-2} + (1-a)^{p-2}]>0$$

and $\partial g/\partial a = 0$ for $a = 1/2$ where g is given as in the foregoing Lemma. Then we deduce $g(1/2,b)<g(a,b)$, i.e. $2^{1-p}<a^p+(1-a)^p$ for any $a\neq 1/2$ and $p>1$. This means for $p>1$:

$$(1/2\ t_3^{(p)}\ 1/2)=1-2^{1-p/p}>1-[a^p+(1-a)^p]^{1/p} = a\ t_3^{(p)}(1-a),$$

i.e. the thesis.

<u>Theorem 6.7.</u> The t-norms $t_1,t_2,t_3^{(p)}$ with $p>1$ are strictly regu-

lar under complements.

Proof. Since $1/2 \geq a \wedge (1-a) > b \wedge (1-b)$ for $a,b \in [0,1]$, the above Lemmas 6.1 and 6.2 give the thesis.

Now we illustrate some lattice results, defining a total order in $[0,1]$ as

$$a \leqslant_t b \text{ iff } \begin{cases} a\ t(1-a) < b\ t(1-b) \\ \\ a \leqslant b \text{ if } a\ t(1-a) = b\ t(1-b) \end{cases}$$

and two binary operations "$\wedge_t$", "$\vee_t$" as

$$a \wedge_t b = \begin{cases} a & \text{if } a\ t(1-a) < b\ t(1-b) \\ b & \text{if } a\ t(1-a) > b\ t(1-b) \\ a \wedge b & \text{if } a\ t(1-a) = b\ t(1-b), \end{cases}$$

$$a \vee_t b = \begin{cases} a & \text{if } a\ t(1-a) < b\ t(1-b) \\ a & \text{if } a\ t(1-a) > b\ t(1-b) \\ a \vee b & \text{if } a\ t(1-a) = b\ t(1-b) \end{cases}$$

where $a,b \in [0,1]$ and t is any t-norm.

From above, we can show that the structure $\theta^{(t)} = ([0,1], \leqslant_t, \wedge_t, \vee_t)$ is a totally ordered lattice.

We must prove for any $a,b,c \in [0,1]$:

$a \wedge_t a = a$	(idempotency)	(16)
$a \wedge_t b = b \wedge_t a$	(simmetry)	(17)
$a \wedge_t (b \wedge_t c) = (a \wedge_t b) \wedge_t c$	(associativity)	(18)
$a \wedge_t (a \vee_t b) = a \vee_t (a \wedge_t b) = a$	(absorption)	(19)

and dual equalities for the operation "$\vee_t$".

(16) and (17) are immediate. We prove (18) distinguishing several situations:

(i) if $a\ t(1-a)=b\ t(1-b)=c\ t(1-c)$, then $(a\Lambda_t\ b)\Lambda_t\ c=(a\wedge b)\wedge c=a\wedge(b\wedge c)=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

(ii) if $a\ t(1-a)=b\ t(1-b)<c\ t(1-c)$, then $(a\Lambda_t b)\Lambda_t c=(a\wedge b)\Lambda_t c=a\wedge b=a\Lambda_t b=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

(iii) if $a\ t(1-a)=b\ t(1-b)>c\ t(1-c)$, then $a\Lambda_t(b\Lambda_t c)=a\Lambda_t c=c=(a\wedge b)\Lambda_t c=(a\Lambda_t b)\Lambda_t c$, i.e. (18).

Without loss of generality, we suppose always $a\ t(1-a)<b\ t(1-b)$ in what follows.

(iv) if $a\ t(1-a)<b t(1-b)<c t(1-c)$, then $(a\Lambda_t\ b)\Lambda_t\ c=a\Lambda_t c=a=a\Lambda_t(b\wedge c)=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

v) if $a\ t(1-a)<b\ t(1-b)=c t(1-c)$, then $(a\Lambda_t b)\Lambda_t c=a\Lambda_t c=a=a\Lambda_t(b\wedge c)=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

(vi) if $at(1-a)<c\ t(1-c)<bt\ (1-b)$, then $(a\Lambda_t b)\Lambda_t c=a\Lambda_t c=a=a\Lambda_t c=a\Lambda_t(\Lambda_t c)$, i.e. (18).

(vii) if $a\ t(1-a)=ct(1-c)<b\ t(1-b)$, then $(a\Lambda_t b)\Lambda_t c)=a\Lambda_t c=a\wedge c=a\Lambda_t c=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

(viii) if $c\ t(1-c)<a\ t(1-a)<b\ t(1-b)$, then $(a\Lambda_t b)\Lambda_t c=a\Lambda_t c=c=a\Lambda_t(b\Lambda_t c)$, i.e. (18).

Now we show (19). Indeed, if $at(1-a)<bt\ (1-b)$, then $a\Lambda_t(a\Lambda_t b)=a\Lambda_t b=a=ava=aV_t a=aV_t(a\Lambda_t\ b)$. If $at\ (1-a)=bt\ (1-b)$, then $a\Lambda_t(aV_t b)=a\wedge(avb)=a=av(a\wedge b)=aV_t(a\Lambda_t b)$.

Analogous proofs can be presented for the dual properties of (16), (17), (18), (19), corresponding to the operation "V_t".

If t satisfies (15), then $O^{(t)}$ has universal bounds 0 and 1/2 being $0=0\wedge 1<a\wedge(1-a)$ and $b\wedge(1-b)<1/2$ for any $a\neq 0$, $b\neq 1/2$, a,b in [0,1].

The lattice $O^{(t)}$ induces in F(X) a lattice structure defining pointwise for every $A,B \in F(X)$ and $i \in I_n$:

$$A \leqslant_t B \text{ iff } A_i \leqslant_t B_i$$

$$(A \wedge_t B)_i = A_i \wedge_t B_i, \quad (A \vee_t B)_i = A_i \vee_t B_i$$

Note that, if t is strictly regular under complements, the partial ordering $\leqslant_t$ implies the sharpening order defined in (6.3) resulting:

$B \leqslant_{Sh} A \rightarrow B_i \leqslant_{Sh} A_i$ for any $i \in I_n \rightarrow A_i \wedge (1-A_i) \geqslant B_i \wedge (1-B_i)$ for any $i \in I_n \rightarrow A_i\ t(1-A_i) \geqslant B_i\ t(1-B_i)$ for any $i \in I_n$.

If $A_i t(1-A_i) > B_i\ t(1-B_i)$, then $B_i \leqslant_t A_i$. If $A_i t(1-A_i) = B_i\ t(1-B_i)$, it must be $B_i \leqslant A_i \leqslant 1/2$ otherwise if $B_i > A_i \geqslant 1/2$ for some $i \in I_n$, we should have

$$A_i\ t(i-A_i) > B_i\ t(1-B_i),$$

in opposition to the equality.

From now on, for every fuzzy relation $R \in \mathcal{R}$ the functional

$$H_{t,s}(R) = \frac{1}{m} \sum_{j=1}^{m} H_{t,s}({}_jR) = \frac{1}{m} \sum_{j=1}^{m} \{ \underset{k=1}{\overset{P}{s}} [R_{jk}\ t(1-R_{jk})]\} \tag{20}$$

where ${}_jR$ is the fuzzy set on the j-th row of R, t is a t-norm strictly regular under complements and s is a t-conorm verifying the axioms (6.1), (6.2), (6.3), has been assumed as measure of fuzziness of a solution of the equation (3). Related concept of this section can be found in Di Nola and Ventre [17].

7. Characterization of elements of $\mathcal{R}$ with minimal entropy.

Modifying a result of Di Nola-Sessa [15], we dirst present this lemma:

Lemma 7.1. If t satisfies (16) and $a \leqslant b \leqslant c$, then

$$b\ t(1-b) \geqslant (a\Lambda_t c)t(1-(a\Lambda_t c)).$$

Proof. We distinguish two cases:

(j) if $a\wedge(1-a) \leqslant c\wedge(1-c)$, we have $a\wedge(1-a) \leqslant b\wedge(1-b)$ otherwise

$$c\wedge(1-c) \geqslant a\wedge(1-a)>b\wedge(1-b) \tag{21}$$

It must be $b\wedge(1-b)=b$ otherwise (21) implies

$$1-c \geqslant c\wedge(1-c)>1-b$$

in opposition to $b \leqslant c$. Also (21) implies the evident contradiction $a \geqslant a\wedge(1-a)>b$. Therefore form $a\ (1-a) \leqslant c\wedge(1-c)$, we deduce $a\ t(1-a) \leqslant c\ t(1-c)$ and $a\ t(1-a) \leqslant b\ t(1-b)$ by the regularity of t.

If $a\ t(1-a)<c\ t(1-c)$, then $a\Lambda_t\ c=a$.

If $a\ t(1-a)=c\ t(1-c)$, then $a\Lambda_t\ c=a\wedge c=a$.

In both situations, we obtain the thesis.

(jj) if $a\wedge(1-a)>c\wedge(1-c)$, we have $c\wedge(1-c) \leqslant b\wedge(1-b)$ otherwise

$$a\wedge(1-a)>c\wedge(1-c)>b\wedge(1-b) \tag{22}$$

If $b\wedge(1-b)=b$, then (22) implies $a \geqslant a\wedge(1-a)>b$ in opposition to $a \leqslant b$. Therefore $b\wedge(1-b)=1-b$, but (2) gives also a contradiction being $1-c \geqslant c\wedge(1-c)>1-b$ whereas it is $b \leqslant c$. Summarizing, from $a\wedge(1-a)>c\wedge(1-c)$, by the strict regularity of t, we deduce $a\ t(1-a)>c\ t(1-c)$ and $c\ t(1-c) \leqslant b\ t(1-b)$. Being $a\Lambda_t c=c$, then the theorem is completely proved.

Theorem 7.2 [15]. By defining for any $j \in I_m$, $k \in I_r$, $R \in \mathcal{R}$:

$$R^*_{jk} = R_{jk} \Lambda_t \; (Q^{-1} \textcircled{\psi} T)_{jk} \tag{23}$$

the fuzzy relation R^*, whose membership functions are given by (23), belongs to $\mathcal{R}$.

Proof. It follows from Theorem 3.7 being for any $R \in \mathcal{R}$:

$$R \leqslant R^* \leqslant Q^{-1} \textcircled{\psi} T$$

and by definition of the operation "Λ_t".

Theorem 7.3. We have $L^* = L$ for any $L \in \mathcal{L}$.

Proof. Since $L_{jk} \in \{0, (Q^{-1} \textcircled{\psi} T)_{jk}\}$ for any $j \in I_m$, $k \in I_r$, we deduce from the definition "Λ_t":

$$L^*_{jk} = L_{jk} \Lambda_t (Q^{-1} \textcircled{\psi} T)_{jk} = \begin{cases} 0 & \text{if } L_{jk} = 0 \\ (Q^{-1} \textcircled{\psi} T)_{jk} & \text{if } L_{jk} = (Q^{-1} \textcircled{\psi} T)_{jk} \end{cases}$$

and this proves the thesis.

Theorem 7.4. If $\mathcal{R} \neq \emptyset$, then

$$\min_{R \in \mathcal{R}} H_{t,s}(R) = \min_{L \in \mathcal{L}} H_{t,s}(L)$$

Proof. Let $R \in \mathcal{R}$. From Theorem 4.9, there exists a lower solution $L^{(h)} \in \mathcal{M}_h$ such that $L^{(h)} \leqslant R$ for any $h \in I_n$.

But $R \leqslant Q^{-1} \textcircled{\psi} T$ and so $L^{(h)} \leqslant Q^{-1} \textcircled{\psi} T$, i.e. $L^{(h)} \in \mathcal{L}^{(h)}$, $h \in I_n$.

Consequently, the fuzzy relation $L = \bigvee_{h=1}^{n} L^{(h)}$ belongs to $\mathcal{L}$ and it is such that $L \leqslant R \leqslant Q^{-1} \textcircled{\psi} T$. By Lemma 7.1 and Theorem 7.3, we have for every $j \in I_m$, $k \in I_r$:

$$L_{jk} t (1 - L_{jk}) = L^*_{jk} \; t \; (1 - L^*_{jk}) =$$

$$[L_{jk} \Lambda_t (Q^{-1} \textcircled{\psi} T)_{jk}] t [1 - (L_{jk} \Lambda_t (Q^{-1} \textcircled{\psi} T)_{jk})] \leqslant R_{jk} \; t (1 - R_{jk}).$$

Being t strictly regular under complements and s monotone, from (20) we deduce for any $R \in \mathcal{R}$:

$$\min_{L \in \mathcal{L}} H_{t,s}(L) \leqslant H_{t,s}(L) \leqslant H(R)$$

and so the theorem is proved.

Another characterization's theorem appears in [10], [30].

Example 7. By putting $t=t_2$ and $s=s_3^{(1)}$, from Theorems 6.1, 6.2, 6.3, 6.4, 6.7, the functional $H_{t_2,s_3^{(1)}}$ defined by (20) satisfies the axioms 6.1, 6.2, 6.3 of De Luca and Termini.

Reconsidering example 6, we have:

$\min_{R \in \mathcal{R}} H_{t_2,s_3^{(1)}}(R) = \min\{H_{t_2,s_3^{(1)}}(L_3), H_{t_2,s_3^{(1)}}(L_4)\} = \min\{0.206, 0.153\} = 0.153$

being $H_{t_2,s_3^{(1)}}({}_1L_3) = s_3^{(1)}(0.21, s_3^{(1)}(0.09.16)) = 0.46$,

$H_{t_2,s_3^{(1)}}({}_1L_4) = s_3^{(1)}(0.21, s_3^{(1)}(0.09,0)) = 0.3$, $H_{t_2,s_3^{(1)}}({}_2L_3) =$

$= H_{t_2,s_3^{(1)}}({}_2L_4) = 0, H_{t_2,s_3^{(1)}}({}_3L_3) = H_{t_2,s_3^{(1)}}({}_3L_4) =$

$s_3^{(1)}(0, s_3^{(1)}0, 0.16)) = 0.16$ and $m=3$.

All the fuzzy relations which have minimal entropy measure equal to 0.153 are the following: L_4 and

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 1.0 & 1.0 \\ 0.0 & 1.0 & 0.8 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 1.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.8 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 0.8 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 0.8 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{pmatrix} 0.7 & 0.9 & 0.0 \\ 1.0 & 1.0 & 1.0 \\ 0.0 & 1.0 & 0.8 \end{pmatrix} \end{array}.$$

8. Approximate solutions.

In the considerations presented until now, we have assumed that the family M (resp. R) of solutions of eq. (4) (resp. (3)) is nonempty. Of course it may happen in some applications that this assumption is too restrictive and may be violated. In such a situation two general ways are proposed:

(P_1) use the formulas involving the operator and evaluate the "fitness" of the obtained results with respect to the right-hand fuzzy relation of fuzzy set.

(P_2) an essence of a constructive way lies in determination of the fuzzy set or relation that minimizes a given performance index (performance criterion).

In order to follow (P_1) let us remember the possibility measure concept of Zadeh [58]:

<u>Definition 8.1.</u> Let $A, A' \in F(X)$. A possibility measure of A with respect to A', under a t-norm, the number Poss(A/A') of [0,1]

given by

$$\text{Poss}\ (A/A') = \bigvee_{i=1}^{n} (A_i\ t\ A'_i)$$

This number expresses the grade of intersection of A and A'. If $A_i = A'_i = 1$ for some $i \in I_n$, i.e. the fuzzy sets A and A' are both normal and further the maximal value of the membership function coincides in the same point, then the possibility measure is equal to 1.

For A and A' disjoints, i.e. $A_i \wedge A'_h = 0$ for every $i, h \in I_n$, we have Poss $(A/A') = 0$.

Therefore in the evaluation of the fuzzy set (relation) as the approximate solution of the equation, calculate:

- for the eq. (3) : poss $((Q^{-1} \circledS T) \square_t Q/T)$

- for the eq. (4) : poss $((A \circledS B) \square_t A/B)$

A different approach to the evaluation of solutions of fuzzy relation equations has been introduced by Gottwald [25]. This approach is based on a slight different interpretation of Theorems of section 3. We first recall some definitions.

<u>Definition 8.2.</u> For $A, A' \in F(X)$ the statement "A is contained in A' (in sense of the t-norm)" has a truth value equal to

$$\tau(A \subset_t A') = \bigwedge_{i=1}^{n} (A_i\ \psi\ A'_i),$$

where ψ corresponds to the t-norm.

<u>Definition 8.3.</u> For $A, A' \in F(X)$, the statement "A and A' are equivalent (in sense of the t-norm)" has a truth value equal to

$$\tau(A \equiv_t A') = [\tau(A \subset_t A')]\ t\ [\tau(A' \subset_t A)].$$

Then the Theorem 3.4 is reformulated as

Theorem 8.1. For the equation (3), the following equality holds:

$$\tau(R \,\square_t\, Q \equiv_t T) = \tau[Q^{-1} \textcircled{ψ} (R \,\square_t\, Q) \equiv_t T]$$

Proof. It is based on the Theorems 3.2 and 3.3 interpreted in terms of truth values of fuzzy relations and it is omitted.

The truth value of $Q^{-1} \textcircled{$\psi$} (R \,\square_t\, Q) \equiv_t T$ is viewed as an index pointing out the solvability degree of the eq. (3). If this equation has solutions, then this solvability degree is 1 and if this is smaller than 1, the equation has no solutions. The best (in the sense of "$\equiv_t$") possible approximations of T by $R \,\square_t\, Q$ have a truth value for $R \,\square_t\, Q \equiv_t T$ not greater than this degree.

Example 8. Let n=2, m=3 and, considering the eq. (4) with t specified as t_1, let be $M \in F(X \times Y)$, $B \in F(Y)$ given by

$$M = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1.0 & 0.6 & 0.4 \\ 0.7 & 0.5 & 0.2 \end{pmatrix} \end{array} \qquad B = \begin{array}{ccc} y_1 & y_2 & y_3 \\ [0.9 & 0.7 & 0.4] \end{array}$$

Hence

$$\hat{A} = M \textcircled{ψ_1} B = \begin{array}{cc} x_1 & x_2 \\ [0.9 & 1.0] \end{array}$$

By straightforward computations one has

$$M \,\square_{t_1}\, \hat{A} = \begin{array}{ccc} y_1 & y_2 & y_3 \\ (0.9 & 0.6 & 0.4) \end{array} \neq B$$

Further

$$(M \,\square_{t_1}\, \hat{A}) \subset_{t_1} B = \bigwedge_{j=1}^{3} [(M \,\square_{t_1}\, \hat{A})(y_j) \psi_1 B(y_j)] = 1$$

$$M \subset_{t_1} (M \,\square_{t_1}\, \hat{A}) = \bigwedge_{j=1}^{3} [B(y_j) \psi_1 (M \,\square_{t_1}\, A)(y_j)] = 0.6$$

Thus the solvability index is equal to 0.6.

In order to follow (F_2), we remind the reader to the papers [6], [20], [44].

9. Some applicational aspects.

One of the advantages of the discussion of the fuzzy relation equations lies in fact that many topics considered in fuzzy set theory may be fruitfully formulated and generalized in the proposed framework of this class of equations. Let us present briefly some representative fields of applications.

9.1. Fuzzy system analysis.

The fuzzy system as introduced by Zadeh [56] has a strong motivation in the fact that, in the real world, there is a great number of systems with a factor of uncertainty, which may occur in the structure of the system (viz. the relationships between inputs and outputs are not precisely defined) or the inputs are fuzzy indeed. These two situations may be described by fuzzy relation equations.

A fuzzy system of the first order takes a form of this type [43]:

$$A_{h+1} = B_h \,\square_t\, A_h \,\square_t\, M$$

where $A_h \in F(X)$, $B_h \in F(Y)$, $M \in F(X \times Y \times X)$, being B_h fuzzy sets of inputs (control) and A_h, A_{h+1} fuzzy sets of state in discrete time moments.

A main problem arising in the proposed model is concerned with the determination of the relation M. For the fixed triangular norm and the fixed order of the equation, it is a particular identification problem, discussed in Hirota and Pedrycz [31], [32].

9.2 Decision - making in fuzzy environment.

Generally speaking, in one-step decision-making we are faced with a set of goals and constraints defined in appropriate spaces:

$G_1, G_2, \ldots G_n$ - fuzzy goals $G_i \in F(\mathbb{G}_i)$

$C_1, C_2, \ldots C_m$ - fuzzy constraints $G_j \in F(\mathbb{C}_j)$

where $i \in I_n$, $j \in I_m$.

A decision state, that is performed with regard to the fuzzy sets above listed, is treated as a fuzzy one D belonging to the space of decisions $\mathbb{D}$.

Here it is plausible to handle the relationships between goals, constraints and decisions in the form of the fuzzy relation M defined in

$$\mathop{\times}_{i=1}^{n} \mathbb{G}_i \times \mathop{\times}_{j=1}^{m} \mathbb{C}_j \times \mathbb{D}.$$

Then the model is put down [42] as follows:

$$D = G_1 \,\square_t\, G_2 \,\square_t \ldots \square_t\, G_n \,\square_t\, C_1 \,\square_t\, C_2 \,\square_t \ldots \square_t\, C_m \,\square_t\, M \tag{24}$$

which is, obviously, an extensive version of the eq. (4).

The model is constructive in such a way that it permits us to

- build the fuzzy relation while analyzing some decision situations (viz. a collection of goals, constraints and decision proposed).
- evaluate grades of importance of various goals and constraints.

We stress that the Bellman-Zadeh [3] scheme of decision-making forms a special case of this model.

Remembering that statistical pattern calssification depends heavily on building decision-making devices, one may easily consider (24) as a formal model of pattern classifier in the case of fuzzy form of undertainty.

An interesting survey in this area is due to Kickert [33].

9.3. On solving equations with fuzzy numbers.

It is remarkable to notice that equations with fuzzy numbers may be immediately solved via methods of fuzzy relation equations [39].

We consider, for instance, addition of fuzzy numbers:

$$A \oplus M = B \tag{25}$$

If A and B are given, we are looking on M, being A,B,M fuzzy sets defined in the set of reals R. The answer is not obvious at the first glance due to the fact the $A + (-A) = 0$ does not hold (-A denotes the inverse of A as $(-A)(x) = A(-x) \in [0,1]$ for any $x \in R$). It means that M cannot be expressed as B - A Rewriting (2) with the use of membership functions, we have

$$B(b) = \bigvee_{\substack{a,x \in R \\ b=a+x}} [A(a) \text{ t } M(x)]$$

$b \in R$. Further, we introduce the relation $A_\oplus$

$$A_\oplus(x,b) = A(b-x)$$

which gives

$$B(b) = \bigvee_{x \in R} [M(x) \text{ t } A_\oplus(x,b)] = (A_\oplus \square_t M)(b).$$

It yields the fuzzy number $A_\oplus \circledpsi B$ fulfilling (25) as the greatest fuzzy number.

For the crisp intervals, ψ reduces to an operator known as Minkowki substraction ")+(" [20].

The same approach can be applied for different operations on fuzzy numbers.

For further applications, we refer to [8], [21], [22], [45].

10. Concluding comments.

In a most general setting, it could be discussed a fuzzy equation of the type

$$\mathop{s}_{j=1}^{m} [Q(x_i, y_j) \ t \ R(y_j, z_k)] = T(x_i, z_k) \tag{26}$$

where $i \in I_n$, $j \in I_m$, $k \in I_r$, t is any t-norm and s is any t-conorm.

We guess that the equation (26) cannot be treated as the equation (3). Indeed, a difficult question arises in the consideration of the operator ψ_t, which must be associated to the equation (26).

We remember (cfr. Section 2) that

$$a \ \psi_t \ b = \sup \{x \in I_t(a,b)\} \tag{27}$$

and it is not possible, extending the formula (27), to substitute the "sup" with any t-conorm s which acts only on finite subsets of [0,1], whereas the particular t-conorm "sup" acts on $I_t(a,b)$, which is an infinite subset of [0,1]. In forthcoming papers, we shall try deeper investigations on the equation (26).

Acknowledgements. We are very grateful to an anonymous referce, whose valuable comments have improved a first version of this survey.

We also thank D. Dubois, S. Gottwald, M. Higashi, E.P. Klement, G. J. Klir, H. Prade, E. Sanchez, M. Sugeno, E. Trillas, R. R. Yager for providing us with reprints of their papers.

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