STOCHASTICA Vol. VIII, Nº 1, 1984

AN ALGORITHM OF CALCULATION OF LOWER SOLUTIONS OF FUZZY RELATIONAL EQUATIONS

Antonio Di Nola

ABSTRACT

The paper deals with the problem of the determination of lower solutions of fuzzy relational equations. An algorithm of calculation of such a solution is presented.

1. Introduction.

An extension of well known Boolean equations was provided by Sanchez [4] by introducing the fuzzy relational equations in order to describe fuzzy systems by means fuzzy relations.

In [4] and [5] the method of resolution of fuzzy relational equations was given.

In [1], [2], [3] and [6] the authors obtained further theoretical results dealing e.g. with lower solutions of the fuzzy equations and their algebraic characterization. In this paper we present an algorithm of calculation of lower solutions of fuzzy relational equations.

2. Basic definitions and notations.

Let L denote a fixed totally ordered lattice with universal bounds 0 and 1. Let's denote by "$\wedge$" and "$\vee$" the meet and join operations in L, by "$\leqslant$" the ordering of L.

Let X be a nonempty set and $F(X) = \{A: X \to L\}$ the set of all fuzzy sets of X [7]. F(X) is a complete lattice with respect to the following relation and operations defined for any $x \in X$:

$$A \leqslant B \quad \text{iff} \quad A(x) \leqslant B(x),$$

$$(A \wedge B)(x) = A(x) \wedge B(x),$$

$$(A \vee B)(x) = A(x) \vee B(x),$$

where $A, B \in F(X)$.

Definition. A fuzzy relation R between two nonempty sets X and Y is an element of F(XxY). We define inverse of R the fuzzy relation $R^{-1} \in F(X \times Y)$ as:

$$R^{-1}(y,x) = R(x,y) \quad \text{for any } (x,y) \in (X \times Y).$$

Definition. Let be $Q \in F(X \times Y)$ and $R \in F(Y \times Z)$ two fuzzy relations. We define the fuzzy relation $T \in F(X \times Z)$, $T = RoQ$ as:

$$(RoQ)(x,z) = \bigvee_{y \in Y} [Q(x,y) \wedge R(y,z)] \qquad \forall (x,z) \in (X \times Z).$$

Definition. We define the fuzzy relation $U \in F(X \times Z)$ as:

$$U(x,z) = (Q \circled{\alpha} R)(x,z) = \bigwedge_{y \in Y} [Q(x,y) \ \alpha \ R(y,z)] \qquad \forall (x,z) \in (X \times Z)$$

where

$$Q(x,y)\ \alpha\ R(y,z) = \begin{cases} 1 & \text{if } Q(x,y) \leqslant R(y,z) \\ R(y,z) & \text{if } Q(x,y) > R(y,z). \end{cases}$$

Let suppose that $X = \{x_1,\dots,x_n\}$, $Y = \{y_1,\dots,y_m\}$ $Z = \{z_1,\dots,z_p\}$ are finite sets, we put $I_n = \{1,\dots,n\}$ the set of first n natural numbers and $I_0 = \Phi$. Furthermore we put: $\forall\ i \in I_n,\ j \in I_m,\ k \in I_p$

$$Q(x_i,y_j) = q_{ij}$$

$$T(x_i,z_k) = t_{ik}$$

$$R(y_j,z_k) = r_{jk}.$$

Now let's consider the fuzzy relational equation

$$RoQ = T \tag{1}$$

where R is an unknown fuzzy relation. Let $\mathcal{R}$ denote the set of all solutions of Eq. 1, and let's suppose that $\mathcal{R} \neq \emptyset$, then the relation $S = Q^{-1}\ \alpha\ T$ is the greatest element of $\mathcal{R}$ (see [4]). From now on we suppose that $\mathcal{R} \neq \emptyset$.

3. Lower solutions.

We call lower solution any element L or $\mathcal{R}$ such that the following proposition holds

"for any $R \in \mathcal{R}, R \leqslant L$ implies $R = L$".

Now we establish a theorem for determining a lower solution of Eq. 1.

Let's consider for any $h \in I_m$ and for any $k \in I_p$ the sequences

$$(j(w))_{w \in I_h}$$

of h different elements of I_m such that, by defining the sets

$$B(j(w),s_{wk}) = \{i \in I_n / q_{ij(w)} \wedge s_{wk} = t_{ik}\} \qquad (2)$$

where

$$s_{wk} = \bigvee_{i \in I_n - \bigcup_{w' \in I_{w-1}} [B(j(w'),s_{w'k})]} t_{ik} \qquad (3)$$

for $w \in I_h$, it must be

$$\forall\ i \in I_n \quad \bigvee_{w \in I_h} q_{ij(w)} \wedge s_{wk} = t_{ik} \qquad (4)$$

Let's denote by $\Gamma(h,k)$ the set of the sequences above defined for any $h \in I_m$ and for any $k \in I_p$.

Theorem. For any $k \in I_p$, let $h(k) \in I_m$ be an index such that for any $h' < h(k)$ it is $\Gamma(h',k) = \emptyset$ and $\Gamma(h(k),k) \neq \emptyset$. Furthermore let

$$\gamma(h(k),k) = (j(w))_{w \in I_{h(k)}}$$

and

$$\gamma(h(k),k) \in \Gamma(h(k),k)$$

then we have that the fuzzy relation $R \in F(Y \times Z)$ defined as:

$$\forall\ k \in I_p \qquad \begin{cases} r_{j(w)k} = s_{wk} & \forall w \in I_{h(k)} \\ r_{jk} = 0 & \forall\ j \in I_m - \{j(w)\}_{w \in I_{h(k)}}, \end{cases} \qquad (5)$$

is a lower solution of Eq. 1.

Proof. Let's prove that firstly $R \in \mathcal{R}$. Indeed for any $k \in I_p$

and for any $l \in I_n$ from (4) it follows that there exists and index $w' \in I_{h(k)}$ such that

$$l \in B(j(w'), s_{w'k})$$

and then

$$q_{lj(w')} \wedge s_{w'k} = t_{lk}. \tag{6}$$

Thus from (4) and (6) it follows that

$$\bigvee_{j \in I_m} [q_{lj} \wedge r_{jk}] = \bigvee_{w \in I_{h(k)}} [q_{lj(w)} \wedge s_{wk}] = t_{lk}$$

and this proves that $R \in \mathcal{R}$.

Now let's prove that R is a lower solution. Firstly we observe that from the hypothesis it follows that for any $k \in I_p$ it must be, for any $w, w' \in I_{h(k)}$ such that $w \neq w'$, the sets $B(j(w), s_{wk})$ and $B(j(w'), s_{w'k})$ are not confrontable with respect to the inclusion relation and this implies the following proposition

"$\forall w' \in I_{h(k)} \exists\, l(w') \in I_n$ such that $[q_{l(w')j(w)} \wedge s_{wk}] < t_{l(w')k}$

$\forall\, w \in I_{h(k)}$ and $w \neq w'$. " (7)

Let's suppose that there exists a solution R' of Eq. 1 such that

$$R' < R, \tag{8}$$

then from (5) and (8) it follows that for any $k \in I_p$ it is

$$r'_{jk} = 0 \qquad \forall j \in I_m - \{j(1), \ldots, j(h(k))\} \tag{9}$$

and that there exists an index $\alpha \in I_{h(k)}$ such that

$$r'_{j(\alpha)k} < r_{j(\alpha)k} = s_{\alpha k} \tag{10}$$

furthermore

$$r'_{j(w)k} \leqslant r_{j(w)k} \quad \forall w \in I_{h(k)} \tag{11}$$

By (7) then there exists $1(\alpha) \in I_n$ such that $t_{1(\alpha)k} = s_{\alpha k}$ thus we have that

$$[q_{1(\alpha)j(\alpha)} \wedge r'_{j(\alpha)k}] < t_{1(\alpha)k} , \tag{12}$$

from $t_{1(\alpha)k} = s_{\alpha k}$ and from (3) and (4), it follows that

$$\forall\ w \in I_{h(k)} \text{ such that } w \neq \alpha \quad q_{1(\alpha)j(w)} \wedge s_{wk} < t_{1(\alpha)k}. \tag{13}$$

From (11) and (13) it follows

$$[q_{1(\alpha)j(w)} \wedge r'_{j(w)k}] < t_{1(\alpha)k} \quad \forall\ w \in I_{h(k)} - \{\alpha\} \tag{14}$$

Furthermore from (9), (10), (11), (12) and (14) it follows

$$\bigvee_{j \in I_m} [q_{1(\alpha)j} \wedge r'_{jk}] = \bigvee_{w \in I_{h(k)}} [q_{1(\alpha)j(w)} \wedge r'_{j(w)k}] < t_{1(\alpha)k}. \tag{15}$$

But (15) is in contradiction with $R' \in \mathcal{R}$, then there no exists an index $\alpha \in I_{h(k)}$ satisfying (10) thus (8) does not hold, and then R is a lower solution.

4. An algorithm of calculation of a lower solution.

The above theorem justifies the following algorithm of calculation of a lower solution R of the Eq. 1.

Start

Put $k:=1$ (1)

Else $h(k):=1$ (2)

If $\Gamma(h(k),k)\neq\emptyset$ then $\gamma(h(k),k)\in\Gamma(h(k),k)$

$$\gamma(h(k),k)=(j(w))_{w\in I_{h(k)}} \quad \text{and}$$

$$\begin{cases} r_{j(w)k} = s_{wk} & \forall\ w\in I_{h(k)} \\ r_{jk} = 0 & \forall\ j\in I_m-(\{j(w)\}_{w\in I_h}) \end{cases}$$

if $k < p$ increase $k:= k+1$ go to (1)

if $k = p$ stop.

If $\Gamma(h(k),k) = \emptyset$, increase $h(k)$, $h(k):=h(k)+1$ go to (2).

Example. Find a lower solution of the equation $RoQ = T$ where

$$Q = \begin{pmatrix} .3 & .7 & .6 & .1 \\ .5 & .8 & .2 & .4 \\ .6 & .3 & .9 & .2 \end{pmatrix} \qquad T = \begin{pmatrix} .6 \\ .4 \\ .6 \end{pmatrix}$$

The algorithm applies and the fuzzy relation

$$R = \begin{pmatrix} 0 & 0 & .6 & .4 \end{pmatrix}$$

is a lower solution.

References.

[1] E. CZOGALA, J. DREWNIAK, W. PEDRYCZ, Fuzzy relation equations on a finite set, Fuzzy Sets and Systems, 7, 1982, 89-101.

[2] A. DI NOLA, A. VENTRE, On booleanity of relational equations in brouwerian lattice, Bollettino dell'Unione Matematica Italiana, to appear.

[3] A. DI NOLA, S. SESSA, On the set of solutions of composite fuzzy relation equationsin totally ordered lattices, Fuzzy Sets and Systems, 9 (1983) 275-285.

[4] E. SANCHEZ, Resolution of composite fuzzy relation equations, Inform. and Control, 30 (1976), 38-48.

[5] E. SANCHEZ, Solutions in composite fuzzy relation equations: application to medical diagnosis in brouwerian logic, in: M. M. Gupta, G. N. Saridis and B. R. Gaines Eds., Fuzzy Automata.

[6] Y. TSUKAMOTO, T. TERANO, Failure diagnosis by using fuzzy logic, Proc. IEEE Conf. Decision Control, New Orleans, 2, 1977, 1390-1395.

[7] L. A. ZADEH, Fuzzy Sets, Inform. and Control 8(1965), 338-353.

Istituto Matematico.
Facoltà di Architettura.
Università di Napoli. Italy .