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ON THE ADDITIVITY OF THE CARDINALITIES OF FUZZY SETS OF TYPE II

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ABSTRACT

In this short note we show that for fuzzy sets of Type II the additive rule for cardinalities holds true. The proof of this result requires an application of approximate reasoning as means of inference by use of the entailment principle.

1. Introduction.

In [1] Zadeh introduces the concept of the sigma count, ΣCount, associated with a fuzzy subset. This count is a crisp measure of the cardinality or the number of elements in a fuzzy subset. In particular if

$$A = \{\frac{a_1}{x_1}, \frac{a_2}{x_2}, \ldots \frac{a_n}{x_n}\}$$

is a fuzzy subset of the set $X = \{x_1 . x_2, \ldots x_n\}$, Zadeh defines the cardinality of A using DeLuca and Termini's [2] concept of the power a fuzzy subset as:

$$\Sigma \text{ Count } (A) = \sum_{i=1}^{n} a_i .$$

In the above a_i is the membership grade of x_i and a_i is considered as a number in the unit interval, $I = [0,1]$.

If A and B are two fuzzy subsets of X the standard or default definition for union and intersection of these two sets are $C=A \cup B$, where $C(x) = \text{Max}[A(x),B(x)]$ and $D = A \cap B$, where $D(x) = \text{Min}[A(x),B(x)]$ ([3]).

In [4] Zadeh shows that using these definitions for intersection and union the following relationship is true

$$\Sigma \text{ Count } (A \cup B) = \Sigma\text{Count } (A) + \Sigma\text{Count } (B) - \Sigma\text{Count } (A\cap B).$$

We shall be interested here in investigating the validity of this equation for fuzzy subsets of Type II, recalling that fuzzy subsets of Type II are those in which the membership grades themselves are fuzzy subsets of the unit interval.

Before proceeding to our investigation we must introduce an important concept from Zadeh's theory of approximate reasoning [5]. This concept, called the entailment principle, provides a very significant tool for making inferences from information. The entailment principle can be stated as follows:

Assume V is a variable taking values in the set Y. Let A and B be two fuzzy subsets of Y such that $A \subset B$. The entailment principle states that from the datum: V is A, we can infer that V is B is a valid statement. The entailment principle is a reflection of our ability to go from specific facts to less specific statements. This principle is examplified by the ability to go from the statement John is 30 years old to the statement John is over 24 years old.

2. Cardinality of fuzzy subsets of type II.

Assume A is a Type II fuzzy subset of the finite set X we define the value of the variable the cardinality of A, denoted Σ Count (A), as

$$\Sigma \text{ Count } (A) = \sum_{i=1}^{n} a_i$$

where a_i, the membership grade of x_i in A, is a fuzzy subset of I. In order to calculate the above sum we need to introduce fuzzy arithmetic ([6]). Assume G and H are two subsets of the real line and # is any binary arithmetic operation: addition, subtraction, multiplication, division, maximum or minimum. We define: $E = G \,\#\, H$ such that E is also a fuzzy number in which, for each $z \in R$, $E(z) = \text{Max}\{[G(x) \wedge H(y)] \mid x, y \in R \text{ such that } z = x \# y\}$.

We note that if A and B are two Type II fuzzy subsets of X then the standard union and intersection operations become $C = A \cup B$ where for each $x \in X$ $C(x) = A(x) \vee B(x)$ and $D = A \cap B$ satisfies $D(x) = A(x) \wedge B(x)$.

In these cases $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are fuzzy subsets of I.

Before proceeding to our results on the cardinality of fuzzy subsets of Type II we shall prove a lemma which will be useful.

<u>Lemma.</u> Assume R, S, T, U are fuzzy numbers such that $R \subset S$ and $T \subset U$ then $R \oplus T \subset S \oplus U$.

Proof. Let $G = R \oplus T$ and $H = S \oplus U$ then for any $z \in$ Real $G(z) = \text{Max}\{[R(x) \wedge T(y)],\ x, y \in \text{Real},\ x + y = z\}$, $H(z) = \text{Max}\{[S(x) \wedge U(y)],\ x, y \in \text{Real},\ x + y = z\}$.

Since $R \subset S$ implies $R(x) \leqslant S(x)$, for all x, and $T \subset U$ implies $T(y) \leqslant U(y)$ for all y, then for any S, y, such that $x + y = z$, $R(x) \wedge T(y) \leqslant S(x) \wedge U(y)$ and the result follows. ∎

Theorem 1. Assume A and B are two Type II fuzzy subsets of X then

$$\Sigma \text{ Count } (A \cup B) \subset \Sigma \text{ Count } (A) + \Sigma \text{ Count } (B) - \Sigma \text{ Count } (A \cap B).$$

Proof. Let $K = A \cup B$ where K_i, A_i, B_i shall denote the fuzzy subsets of I representing the membership grades of x_i in K, A and B respectively. Thus $K_i = A_i \text{ⓥ} B_i$ and hence for any $z \in I$ $K_i(z) =$ $= \text{Max}\{[A_i(u) \wedge B_i(y)],\ y,\ u \in I, \text{ such that } u \vee y = z\}$,

$\Sigma \text{ Count } (K) = \sum_{i=1}^{n} K_i$.

Let $M = A \cap B$, then $M_i = A_i \text{Ⓐ} B_i$. Let Σ Count (A) + ΣCount (B) - ΣCount $(A \cap B) = \sum_{i=1}^{n} P_i$ where $P_i = A_i \oplus B_i \ominus M_i$.

For any $z \in I$, $P_i(z) = \text{Max}\{[A_i(a) \wedge B_i(b) \wedge A_i(c) \wedge B_i(d)],$ $a,b,c,d \in I$, such that $a+b-(c \wedge d) = z\}$.

Let u^* and y^* be such that $u^* \vee y^* = z$ and $K_i(z)=A_i(u^*)\wedge B_i(y^*)$.

If we let $a = c = u^*$ and $b = d = y^*$, then $u^* + y^*-(u^* \wedge y^*)=$ $=u^* \vee y^* = z$, hence $P_i(z) \geqslant A_i(u^*)\wedge B_i(y^*)\wedge A_i(u^*)\wedge B_i(y^*) \geqslant$ $\geqslant A_i(u^*) \wedge B_i(y^*) \geqslant K_i(z)$.

Thus since this is valid for all z, $K_i \subset P_i$ from our previous lemma $\sum_{i=i}^{n} K_i \subset \sum_{i=1}^{n} P_i$, and the theorem follows. ■

Theorem 2. Assume A and B are two Type II fuzzy subsets of X then

$$\Sigma \text{ Count } (A \cup B) \text{ is } \Sigma \text{ Count } (A) + \Sigma \text{ Count } (B) - \Sigma \text{ Count } (A \cap B)$$

is a true statement

Proof. We shall denote the ΣCount $(A \cup B)$ as the variable V. V can be said to be equal to some quantity D whose value is unknown, V is D. Let us denote the quantity ΣCount (A) + ΣCount (B)- Σ Count $(A \cap B)$ as E. In the previous theorem we have shown that whatever the value D it satisfies the property $D \subset E$. Thus we have V is D and D $\subset$ E we now apply the entailment principle to infer that V is E is a true statement. ■

3. Conclusion.

The essential conclusion of this theorem is to state that the possibility distribution [7] associated with the variable Σ Count (A $\cup$ B) can be obtained by adding the value of the sigma Count of A and the sigma Count of B and subtracting the sigma Count of A $\cap$ B. We note that while the exact value of ΣCount (A$\cup$B) is $\sum_{i=1}^{n} a_i \textcircled{v} b_i$, in providing ΣCount (A $\cup$ B) as

Σ Count (A) + Σ Count (B) - Σ Count (A $\cap$ B) we are providing the value for Σ Count (A $\cup$ B) in a less specific manner in the same sense as saying John is over 25 years old when in fact he is 30 years old.

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