STOCHASTICA Vol. VII, Nº 3, 1983

ON SOME ISOMORPHISME OF DE MORGAN ALGEBRAS OF FUZZY SETS

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0. Introduction.

In this paper are studied the classes of De Morgan Algebras $(\underset{\sim}{P}(X),\cap,\cup,n)$.

With respect to isomorphisms of such algebras, being $\underset{\sim}{P}(X)$ the fuzzy sets on a universe X taking values in [0,1], $\cup$ and $\cap$ the usual union and intersection given by max and min operations and n a pro per "complement".

So, we start from the existing results on negations in ordered sets and lattices 3, 12, on negation functions in [0,1] 6, 7, 9, 11, on negations fulfiling either the extension principle or the generalized extension principle in fuzzy sets 10, 12, 13, from results on De Morgan Algebras 4, and, particularly, those concerning to De Morgan Algebras of fuzzy sets 2, 5 and from results for functionally expressable morphisms between De Morgan Algebras of fuzzy sets 8, 11.

Following this line we obtain that only De Morgan Algebras defined in universes of the same cardinal can be of the same class and that, given a universe, there are as many classes as conjugation classes of involutive permutations of the given universe.

We denote by L(X) the lattice $(\underset{\sim}{P}(X),\cap,\cup)$ of fuzzy sets on the universe X, being $\cap$ and $\cup$ defined by min and max respectively, by P(X) the Boolean Algebra of crisp subsets of X, by C the unique complement of P(X), by $\bar{A}$ the complement of $A \in P(X)$ obtained by C, by δ_x the singleton of P(X) defined by $\delta_x(x)=1$ and $\delta_x(a)=0$ when $a \neq x$, by $\underset{\sim}{\alpha}$ the constant fuzzy set equal to $\alpha \in [0,1]$ and by B[0,1] and N[0,1] respectively the set of increasing bijections from [0,1] to [0,1] and the strong negation functions on [0,1].

1. Automorphisms of fuzzy sets lattice L(X).

Definition 1. An automorphisms H_s of L(X) is said to be an automorphism generated by a permutation s of X if it is defined by, $H_s(A)=A \circ s$ for all $A \in \underset{\sim}{P}(X)$.

Definition 2. An automorphism H of L(X) is said to be pointwise functionally expressable (p.f.e. from now on) for the family $\{f_x \in B[0,1] \mid x \in X\}$ if it is defined by, $(H(A))(x)=f_x(A(x))$ for all $x \in X$ and $A \in P(X)$.

In the particular case of being all f_x equal to the same $f \in B[0,1]$ we say that the automorphism H is functionally expressable (f.e. from now on) from function f and the definition of H will be, $H(A) = f \circ A$ for all $A \in P(X)$.

Obviously H_s given by def. 1 and H given by def. 2 are automorphisms and so they are well defined.

Proposition 1. Any automorphism H of L(X) is extension of an automorphism of P(X).

Proof. For any $A \in P(X)$ it is $\emptyset = H(\emptyset) = H(A \cap \bar{A}) = H(A) \cap H(\bar{A})$ and $X = H(X) = H(A \cup \bar{A}) = H(A) \cup H(\bar{A})$. Then for any $x \in X$ it is $H(A)(x) \wedge H(\bar{A})(x) = 0$ and $H(A)(x) \vee H(\bar{A})(x) = 1$, so $\{H(A)(x), H(\bar{A})(x)\} = \{0,1\}$ for any $x \in X$, which implies $H(A) \in P(X)$ and as a consequence, $H|_{P(X)}$ is an automorphism of P(X).

Proposition 2. Any automorphism of P(X) is of the type H_s for a certain permutation s of X.

Theorem 1. Any automorphism H of L(X) can be obtained by composition of an automorphism H_s generated by a permutation s of X with a p.f.e. automorphism. Such descomposition of H is unique.

Proof. $H|_{P(X)}$ is an automorphism of P(X), then from prop. 2 it is an automorphism generated by a permutation s of X. Let $H' = H \circ H_{s^{-1}}$. Clearly H' is an automorphism of L(X) by which any $A \in P(X)$ are kept fixed as

$$H'(\delta_x) = H \circ H_{s^{-1}}(\delta_x) = H(\delta_{s^{-1}(x)}) = \delta_{s(s^{-1}(x))} = \delta_x.$$

So, H', for any $x \in X$, is an increasing bijection from $[\emptyset, \delta_x]$ to itself, being

$$[\emptyset, \delta_x] = \{A \in \underset{\sim}{P}(X) \mid \emptyset \leqslant A \leqslant \delta_x\} = \{\delta_x \wedge \underset{\sim}{\alpha} \ \alpha \in [0,1]\}.$$

Then, for any $x \in X$, H' define a $f_x \in B[0,1]$ given by $f_x(\alpha) = (H'(\delta_x \wedge \underset{\sim}{\alpha}))(x)$. Clearly H' is the automorphism generated by the family $\{f_x \in B[0,1] \mid x \in X\}$. So, $H = H' \circ H_s$, and because of being s univocally defined, as it is a permutation such that $H_{P(X)} = H_s$, then the descomposition is unique.

2. De Morgan Algebras of Fuzzy Sets.

Let L(X) be the lattice of fuzzy sets on X. In order to get a De Morgan Algebra it is necessary to define a "complement" such that adequate properties are fulfilled. In 3, 6, 7, 11 it is clear that such "complement" should be defined by an strong negation or involution $n: \underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ such that n is non-increasing, involutive, $n(\emptyset) = X$ and $n(X) = \emptyset$.

Now we will further study such strong negations.

<u>Proposition 3.</u> If n be a strong negation on $\underset{\sim}{P}(X)$, then $n|_{P(X)}$ is a strong negation on P(X).

Proof. If $A \in P(X)$, then it is $\emptyset = n(X) = n(A \cup \bar{A}) = n(A) \cap n(\bar{A})$ and $X = n(\emptyset) = n(A \cap \bar{A}) = n(A) \cup n(\bar{A})$, and this implies that $n(A) \in P(X)$.

Then it is necessary to study strong negations n on $\underset{\sim}{P}(X)$ that are extensions of the strong negations of the lattice P(X). Such negations are completely described in 12, 13. In order to give the obtained results in this line, we start from results on negation functions given in 6, 7, 11 and we recall the following definition:

<u>Definition 3.</u> A negation n on $\underset{\sim}{P}(X)$ is said to be pointwise functionally expressable (p.f.e. from now on) from a family of negation functions $\{n_x \in N[0,1] \mid x \in X\}$ if for any A, n(A) is the fuzzy set given by,

$$(n(A))(x) = n_x(A(x)) \text{ for any } x \in X.$$

Strong negations such that give a De Morgan Algebra structure to the set of all fuzzy sets are of two types:

A.- Negations such that fulfil the extension principle 10, 12, 13 (E.P. from now on), that is, negations n such that $n|_{P(X)}=C$. Such negations are characterized by the following result: "A negation n on $\underset{\sim}{P}(X)$ fulfils the E.P. iff it is p.f.e."

B.- Negations such that fulfil the generalized extension principle 12, 13 (G.E.P. from now on), that is, negations n such that $n|_{P(X)}$ is a strong negation on P(X).

Such negations are characterized by the following result: "A negation n on $\underset{\sim}{P}(X)$ fulfils the G.E.P. iff it is obtained by composing an automorphism generated by a permutation s of X such that $s^2=I$ with a p.f.e. negation n' from a family $\{n'_x \in N[0,1] \mid x \in X\}$ such that $n'_x = n'_{s(x)}$ for any $x \in X$.

Remarks. 1) The descomposition $n = n' \circ H_s$ is unique and n' is called the p.f.e. negation associated to n.

2) If s_{n_x} is the fixed point of the negation function n_x, the simmetry level of a strong negation n is the fuzzy set s_n defined by $s_n(x) = s_{n'_x}$ being n' the p.f.e. negation associated to n.

3. Classes of De Morgan Algebras of Fuzzy Sets.

Now we are going to study the classes of De Morgan Algebras of fuzzy sets on X with respect to isomorphisms of such algebras.

Recall that $H: (P(X),\cap,\cup,n) \longrightarrow (P(X),\cap,\cup,n')$ is an isomorphism of the algebra if it is a bijection and it satisfies:

a) $H(A \cup B) = H(A) \cup H(B)$ for any $A,B \in P(X)$,

b) $H(A \cap B) = H(A) \cap H(B)$ for any $A,B \in P(X)$,

c) $H(n(A)) = n'(H(A))$ for any $A \in P(X)$.

As negations we are going to use come from negation functions we will recall the following notation and result 8, 11:

"Given a strong negation function n on [0,1] we denote by N_n, P_n and s_n the negative, positive and fixed point of n, that is, $N_n = \{x \in [0,1] \mid n(x)>x\}$, $P_n = \{x \in [0,1] \mid n(x)<x\}$ and $n(s_n)=s_n$".

"Given two strong negation functions n and n' on [0,1] the unique increasing and bijective functions $h:[0,1] \to [0,1]$ that commute with n and n' ($h\circ n=n'\circ h$) are of the type h_γ defined by:

$$(1) \qquad h_\gamma(x) = \begin{cases} \gamma(x) & : \text{ if } x \in N_n, \\ s_{n'} & \text{ if } x = s_n, \\ n'(\gamma(n(x))) & \text{ if } x \in P_n, \end{cases}$$

being γ an increasing bijection from N_n to $N_{n'}$".

Theorem 2. The De Morgan Algebras $(\underset{\sim}{P}(X),\cap,\cup,n)$ such that n satisfies the E.P. constitutes an equivalence class.

Proof. Let n and n' be negations on $\underset{\sim}{P}(X)$ such that satisfy the E.P. and $\{n_x \in N[0,1] \mid x \in X\}$, $\{n'_x \in N[0,1] \mid x \in X\}$ be the families of negation functions that define n and n' respectively. It is easy to verify that $(\underset{\sim}{P}(X),\cap,\cup,n)$ is isomorphic to $(\underset{\sim}{P}(X),\cap,\cup,n')$ by means of the p.f.e. isomorphism H from functions $\{h_{\gamma_x} \in B[0,1] \mid x \in X\}$ given by:

$$h_{\gamma_x}(a) = \begin{cases} \gamma_x(a) & \text{if } a \in N_{n_x}, \\ s_{n'_x} & \text{if } a = s_{n_x}, \\ n'_x(\gamma_x(n_x(a))) & \text{if } a \in P_{n_x}, \end{cases}$$

being γ_x an increasing bijection from N_{n_x} to $N_{n'_x}$.

On the other hand it is clear that, if n satisfies the E.P. and $\bar{n}$ does not, then $(\underset{\sim}{P}(X),\cap,\cup,n)$ is not isomorphic to $(\underset{\sim}{P}(X),\cap,\cup,\bar{n})$ as any isomorphism of algebras preserves complements and transforms P(X) in itself, and $(P(X),\cap,\cup,n|_{P(X)})$ is a Boolean Algebra but $(P(X),\cap,\cup,\bar{n}|_{P(X)})$ is not.

This last result says the De Morgan Algebras of fuzzy sets on a given universe X such that contain the Boolean Algebra P(X) as subalgebra, are isomorphics. Then $(\underset{\sim}{P}(X),\cap,\cup,1-I)$ can be taken as a representant of the class.

In order to study the case of De Morgan Algebras with negations such that verify the G.E.P. it is necessary to recall that any isomorphism H from $\underset{\sim}{P}(X)$ into $\underset{\sim}{P}(X)$ restricted to P(X) is an isomorphism, that any automorphism of P(X) is of the type H_s and that any negation n such that satisfies the G.E.P. is descomposable in the form $n=n'\circ H_s$ with $n|_{P(X)} = C\circ H_s$.

Proposition 4. $H_\sigma:(P(X),\cup,\cap,C\circ H_s) \to (P(X),\cap,\cup,C\circ H_t)$ is isomorphism iff $\sigma\circ s = t\circ\sigma$.

Proof. We have to prove only that for all $x \in X$, it is

$$H_\sigma((C\circ H_s)(\delta_x)) = (C\circ H_t)(H_\sigma(\delta_x)) \qquad (\varepsilon)$$

It is clear that $H_\sigma((C\circ H_s)(\delta_x)) = H_\sigma(C(\delta_{s(x)})) = H_\sigma(\bar{\delta}_{s(x)}) = \bar{\delta}_{\sigma(s(x))}$ and $(C\circ H_t)(H_\sigma(\delta_x)) = (C\circ H_t)(\delta_{\sigma(x)}) = C(\delta_{t(\sigma(x))}) = \bar{\delta}_{t(\sigma(x))}$.
So the necessary and suficient condition for the equality (ε) hold is that $\sigma\circ s = t\circ\sigma$.

<u>Definition 4.</u> Two permutations s and t of X are said to be conjugated iff there exist a permutation σ of X such that $t = \sigma\circ s\circ\sigma^{-1}$.

Clearly such relation is an equivalence. Its classes are called conjugation classes and {I} is one of them.

<u>Theorem 3.</u> $(\underset{\sim}{P}(X),\cap,\cup,n\circ H_s)$ is isomorphic to $(\underset{\sim}{P}(X),\cap,\cup,n'\circ H_t)$, being n and n' p.f.e. negations on $\underset{\sim}{P}(X)$, iff s and t are two conjugated permutations.

Proof. If the isomorphism is satisfied, then prop. 4 show that s and t are conjugated.

If s and t are conjugated we will construct an isomorphism H between both Algebras:

Recall that by theorem 1, it is $H = F\circ H_m$, being F a p.f.e. isomorphism from a family $\{f_x \in B[0,1] \mid x \in X\}$ and m a permutation of X. As $H|_{P(X)} = H_m$, m has to be a permutation, which exist because s and t are conjugated, such that $m\circ s = t\circ m$.

On the other hand we know that $F\circ H_m$ is always an isomophism of the lattices, so it is necessary to choose adequately the family $\{f_x \mid x \in X\}$ in order to get H morphism for the negation. To manage it we have only to impose the condition of being morphism to the elements of the type $\delta_x \wedge \underset{\sim}{\alpha}$.

It is clear that, $(n'\circ H_t)((F\circ H_m)(\delta_x \wedge\underset{\sim}{\alpha}) = (n'\circ H_t)(\delta_{m(x)} \wedge f_{m(x)}(\alpha))$
$= \bar{\delta}_{t(m(x))} \vee n'_{t(m(x))}(f_{m(x)}(\alpha))$,

and that, $(F\circ H_m)((n\circ H_s)(\delta_x \wedge \underset{\sim}{\alpha}) = (F\circ H_m)(\bar{\delta}_{s(x)} v n_{s(x)}(\alpha)) =$

$= \bar{\delta}_{m(s(x)} v f_{m(s(x))}(n_{s(x)}(\alpha))$.

So to get $F\circ H_m$ a isomorphism, it is suficient that $m\circ s = t\circ m$ and to take $f_{m(x)} = f_{m(s(x))}$ as a function of B[0,1] which commutes with $n_{s(x)} = n_x$ and $n'_{t(m(x))} = n'_{m(x)}$. This function always exists as (1) shows.

<u>Corollary 3.1.</u> The number of classes of De Morgan Algebras of fuzzy sets on X is equal to the number of conjugation classes of involutive permutations of X.

It is clear that theorem 3 says that in any class of De Morgan Algebras one of the type $(\underset{\sim}{P}(X), \cap, \cup, (1-I)\circ H_m)$ can be taken as its representative and this algebra has, as symmetry level, the fuzzy $\underset{\sim}{1/2}$.

4. Conjugation classes of involutive permutations.

Let X be a set, G_X the group of permutations of X and I_X the set of involutive permutations, $I_X = \{m \in G_X | m^2 = I\}$.

Clearly for any $m \in I_X$ there are only two types of elements:

a) elements which are fixed by m,

b) elements $a \in X$ such that $m(a) = b \neq a$. In that case $m(b)=a$. Then we can asure that, for any $m \in I_X$, X is descomposed in F_m, set of elements which are fixed by m, and a family P_m of subsets ot two elements {a,b} such that each one corresponds to the other by m.

<u>Proposition 5.</u> Any permutation which is conjugated with one of I_X is of I_X.

Proof. If $m \in I_X$ and $n = s\circ m\circ s^{-1}$, $n^2 = s\circ m\circ s^{-1}\circ s\circ m\circ s^{-1} = s\circ m^2\circ s^{-1} =$
$= s\circ I\circ s^{-1} = I$.

Proposition 6. If $m,n \in I_X$ are conjugated, that is, there exist s such that $n = s \circ m \circ s^{-1}$, then s is a bijection of F_m in F_n and it transforms subsets of P_m in subsets of P_n.

Proof. If $a \in F_m$, then $s(a) \in F_n$ as $n(s(a)) = (s \circ m \circ s^{-1} \circ s)(a) = s(m(a)) = s(a)$. If $b = m(a) \neq a$, then $s(b) = n(s(a))$ as $n(s(a)) = (s \circ m \circ s^{-1})(s(a)) = s(b)$.

Theorem 4. $m,n \in I_X$ are of the same conjugation class iff F_m is coordinable with F_n and P_m with P_n.

Proof. Previous proposition says that the condition is necessary. Conversely, let $s \in G_X$ the permutation which transform F_m in F_n and a pair $\{a,m(a)\}$, with $a \notin F_m$, into a pair $\{b,n(b)\}$ with $b \notin F_n$. Clearly such a satisfy $s \circ m = n \circ s$.

Corollary 4.1. If X is finite, $m,n \in I_X$ are of the same conjugation class iff F_m and F_n have the same number of elements.

Corollary 4.2. If X has n elements the number of conjugation classes of I_X is $(n/2)+1$ if n is even, and $((n-1)/2)+1$ if n is odd.

5. Isomorphisms between algebras of fuzzy sets on different universes.

Let X,Y be two universes and $\underset{\sim}{P}(X)$, $\underset{\sim}{P}(Y)$ the set of fuzzy sets on X and Y respectively with values in [0,1]. A similar proof to proposition 1 allows to state that:

Proposition 7. If $H:\underset{\sim}{P}(X) \to \underset{\sim}{P}(Y)$ is an isomorphism of such lattices, $H|_{P(X)}$ is an isomorphism from P(X) to P(Y).

Corollary 7.1. If L(X) is isomorph to L(Y), then X and Y have the same cardinal.

Proof. Any isomorphism H defines an isomorphism from P(X) to

P(Y) and any isomorphism from P(X) to P(Y) bijectively maps atoms of P(X) into atoms of P(Y). Then a bijection $s:X \to Y$ can be defined as $s(x) = y$ iff $H(\delta_x) = \delta_y$.

<u>Definition 5.</u> Let $s:X \to Y$ a bijection. An isomorphism $H_s:P(X) \to P(Y)$ is said to be an isomorphism defined by s if it is defined by, $H_s(A) = A \circ s$ for any $A \in P(X)$.

<u>Theorem 5.</u> Let X,Y be sets with the same cardinal. Any isomorphism H from the lattice L(X) into L(Y) is composition of an isomorphism H_s generated by a bijection $s:X \to Y$ and an p.f.e. automorphism of $\underset{\sim}{P}(Y)$. Such descomposition is unique.

In order to stablish the classes of De Morgan Algebras of fuzzy sets it seems clear that given the previous results it is necessary to generalize the concept of conjugated involutive permutations.

<u>Definition 6.</u> $s \in I_X$ and $t \in I_Y$ are said to be conjugated iff there exist a bijection $m:X \to Y$ such that $m \circ s = t \circ m$.

Clearly if s,t are conjugated, then X,Y have the same cardinal.

<u>Proposition 8.</u> Let X,Y be two universes such that have the same cardinal. $s \in I_X$ and $t \in I_Y$ are conjugated iff F_s and F_t have the same cardinal and so have P_s and P_t.

<u>Theorem 6.</u> $(\underset{\sim}{P}(X), \cap, \cup, n \circ H_s)$ is isomorph to $(\underset{\sim}{P}(Y), \cap, \cup, n' \circ H_t)$ iff X,Y have the same cardinal and s,t are conjugated.

Proof: If $H:\underset{\sim}{P}(X) \to \underset{\sim}{P}(Y)$ is an isomorphism we already know that X and Y have the same cardinal, that $H = F \circ H_m$ and that $H|_{P(X)} = H_m|_{P(X)} : P(X) \to P(Y)$ is an isomorphism. So $H_m((C \circ H_s)(\delta_x)) = (C \circ H_t)(H_m(\delta_x))$ for any $x \in X$. A similar proof of proposition 4 shows that, in order to be H_m a morphism of algebras it should be $t \circ m = m \circ s$.

On the other hand if X,Y have the same cardinal and s,t are conjugated -there exist m such that $m \circ s = t \circ m$ - H_m is an automorphism from $(P(X),\cap,\cup,n \circ H_s)$ to $(P(Y),\cap,\cup,\bar{n} \circ H_t)$ being $\bar{n}$ the p.f.e. negation defined by the family $\{\bar{n}_{m(x)} \mid \bar{n}_{m(x)} = n_x$ for any $x \in X\}$ and by theorem 3 $(P(Y),\cap,\cup,\bar{n} \circ H_t)$ is isomorph to $(P(Y),\cap,\cup,n' \circ H_t)$ and theorem is proved.

In the particular case that the universe is finite, theorem 6 give the following result.

Corollari 6.1. If the universe has finite cardinal n the number of classes of De Morgan Algebras that can be defined is $(n/2)+1$ if n is even and $((n-1)/2)+1$ if n is odd.

Acknowledgements. I would like to thank Professor Trillas for his constant help and valuable suggestions and also to all my friends in the Department of Mathematics and Statistics. Specialy I wish to thank T. Riera (Facultat d'Informàtica).

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